Math 261y: von Neumann Algebras (Lecture 16)

October 7, 2011

We have seen that every abelian von Neumann algebra A is of the form $L^\infty(X)$, where X is a disjoint union of finite measure spaces. In this lecture, we will study the extent to which X is uniquely determined by A.

Definition 1. By a *finite measure space*, we mean a set X, a σ-algebra Σ of subsets of X (which we will call *measurable sets*, and a countable additive measure $\mu : \Sigma \to \mathbb{R}_{\geq 0}$. In this case, we write Σ_0 for the set $\{K \in \Sigma : \mu(K) = 0\}$ of measurable sets having measure zero.

Let (X, Σ, μ) and (X', Σ', μ') be finite measure spaces. A function $f : X \to X'$ is *measurable* if $f^{-1}K \in \Sigma$ for all $K \in \Sigma'$. We will say that a measurable function f is *measure preserving* if $\mu(f^{-1}K) = \mu'(K)$ for all $K \in \Sigma'$. We will say that f is *quasi-measure preserving* if $\mu'(K) = 0$ implies $\mu(f^{-1}K) = 0$. We will say that a pair of measurable functions $f, f' : X \to X'$ are *equal almost everywhere* if $\{x \in X : f(x) \neq f'(x)\}$ is contained in a measurable set of measure zero.

We define a category $\mathcal{M}$ as follows:

(a) The objects of $\mathcal{M}$ are finite measure spaces (X, Σ, μ) where X is nonempty.

(b) A morphism from (X, Σ, μ) to (X', Σ', μ') is a an equivalence class of quasi-measure preserving functions $f : X \to X'$ (where the equivalence relation is given by "equality almost everywhere").

We let $\mathcal{M}_0$ denote the subcategory of $\mathcal{M}$ whose morphisms are measure preserving maps.

Remark 2. We can allow X to be the empty set in (a) if we are willing to modify (b) a little bit, allowing functions $f : X - K \to X'$ where K is a set of measure zero (we would like $\emptyset$ to be isomorphic in $\mathcal{M}$ to any measure space of total measure zero).

Remark 3. Let (X, Σ, μ) and (X', Σ', μ') be finite measure spaces which are isomorphic in $\mathcal{M}$. Then we have measurable functions $f : X \to X'$ and $g : X' \to X$ which are mutually inverse away from sets of measure zero. Removing sets of measure zero from X and X', we can assume that f and g are mutually inverse bijections. Let us identify X with X' via f. Since f and g are measurable, Σ and Σ' coincide (as σ-algebras of subsets of X). Moreover, since f and g are quasi-measure preserving, the subsets Σ_0 and Σ'_0 also coincide: that is, the measures μ and μ' are absolutely continuous with respect to each other. It follows from the Radon-Nikodym theorem that μ and μ differ by rescaling by an integrable function.

The construction $(X, \Sigma, \mu) \mapsto L^\infty(X)$ determines a functor from the category $\mathcal{M}$ to (the opposite of) the category of abelian von Neumann algebras. We now phrase our basic question as follows: to what extent is this functor invertible? To get a reasonable answer, we need to introduce some hypotheses on our measure spaces.

Definition 4. Let (X, Σ, μ) be a measure space. We say that a measurable subset $K \subseteq X$ is an *atom* if $\mu(K) > 0$ and, for every measurable subset $K' \subseteq K$, either K' or $K - K'$ has measure zero.

If $K, L \subseteq X$ are atoms, then $K \cap L$ is a measurable subset of both K and L. It follows that either $K \cap L$ has measure zero, or the differences $K - K \cap L$ and $L - K \cap L$ have measure zero. In the first case, we will say that K and L are *equivalent*. Up to equivalence, a finite measure space can have at most countable many atoms. For example, if X has measure 1, then it has at most n mutually inequivalent atoms of measure $\frac{1}{n}$. Choose a set of representatives for the atomic subsets $\{K_1, K_2, \ldots\}$. We will denote the union $\bigcup K_i$ by X_a, and refer to it as the *atomic* part of X. It is a measurable subset of X, which is well-defined modulo sets of measure zero. We let $X_c = X - X_a$; we will refer to X_c as the *atomless* part of X. By construction, X_c does not contain any atoms.

Definition 5. Let (X, Σ, μ) be a finite measure space. We will say that (X, Σ, μ) is *standard* if it satisfies the following conditions:

(a) Every atom in X has the form $\{x\} \cup K$, where K is a set of measure zero (in other words, every atom is equivalent to a singleton modulo sets of measure zero).

(b) The atomless part of X is isomorphic (in the category $\mathcal{M}$) to an interval $([0,t], \Sigma_B, \mu_L)$, where $t = \mu(X_c)$, Σ_B is the σ-algebra of Borel subsets of $[0,t]$, and μ_L denotes Lebesgue measure. Here we allow the case $t = 0$ (in case X_c has measure zero).

Remark 6. We will see at the end of this lecture that, if (X, Σ, μ) is standard, the isomorphism required by (b) can be chosen to preserve measure.

Let $\mathcal{M}_s$ denote the full subcategory of $\mathcal{M}$ spanned by those triples (X, Σ, μ) which are standard.

Proposition 7. *The construction $(X, \Sigma, \mu) \mapsto L^\infty(X)$ is fully faithful when restricted to $\mathcal{M}_s$.*

Proposition 7 is a consequence of the following more precise result (and the fact that a von Neumann algebra is determined by the underlying complete Boolean algebra of projections).

Proposition 8. *Let (X, Σ, μ) and (X', Σ', μ') be finite measure spaces, and assume that (X', Σ', μ') is standard. Then the construction*

$$\theta : \mathrm{Hom}_{\mathcal{M}}((X, \Sigma, \mu), (X', \Sigma', \mu')) \to \mathrm{Hom}(\Sigma'/\Sigma'_0, \Sigma/\Sigma_0)$$

is bijective; here the right hand side is computed in the category of complete Boolean algebras.

In what follows, if K is a measurable subset of a measure space (X, Σ, μ), we let $[K]$ denote the equivalence class of K in Σ/Σ_0.

Proof. We first show that θ is injective. Let $f, g : X \to X'$ be quasi-measure preserving functions, and assume that for every measurable subset $K \subseteq X'$, the inverse images $f^{-1}K$ and $g^{-1}K$ differ by a set of measure zero. We wish to show that f and g coincide away from a set of measure zero.

Breaking X' up into its atomic and atomless part, we reduce to the following two cases:

(a) X' is a union of atoms $K_1 \cup K_2 \cup \cdots$. Since X' is standard, we may assume that each of these atoms consists of exactly one point: that is, we have $X' = \{x_1, x_2, \ldots\}$. By assumption, the inverse images $f^{-1}\{x_i\}$ and $g^{-1}\{x_i\}$ agree up to a set of measure zero. Away from the union of these sets, the functions f and g coincide.

(b) X' is atomless. Since X' is standard, it is isomorphic to $[0,t]$ (with Lebesgue measure) for some real number t. For every rational number q, the inverse images $f^{-1}[0,q]$ and $g^{-1}[0,q]$ coincide away from a set of measure zero. Throwing these sets away, we may assume that $f^{-1}[0,q] = g^{-1}[0,q]$ for every rational number, from which it follows immediately that $f = g$ (since the rational numbers are dense).

We now prove that θ is surjective. Suppose we are given a map $\phi : \Sigma'/\Sigma'_0 \to \Sigma/\Sigma_0$; we wish to show that ϕ is induced by a quasi-measure preserving map $f : X \to X'$. As before, we can reduce to two special cases:

(a') X' is atomic, hence we may assume that $X' = \{x_1, x_2, \ldots\}$ where each $\{x_i\}$ has positive measure (and perhaps only finitely many x_is appear). Choose representatives $K_i \subseteq X$ for the sets $\phi[\{x_i\}]$. These sets are disjoint moduli sets of measure zero, and $X - \bigcup K_i$ has measure zero. Modifying our choices by sets of measure zero, we can assume that the K_i are disjoint and that $\bigcup K_i = X$. We can now take f to be the function given by $f(x) = x_i$ if $x \in K_i$.

(b') X' is atomless, hence isomorphic to $[0, t]$ for some $t \geq 0$. If $t = 0$ there is nothing to prove; otherwise, we may rescale and assume $t = 0$. The unit interval $[0, 1]$ is isomorphic (as a finite measure space) to 2^ω (endowed with the product measure), via the map $2^\omega \to [0, 1]$ given by

$$(t_i) = \sum \frac{t_i}{2^{i+1}}.$$

We may therefore assume without loss of generality that $X' = 2^\omega$. For every integer $i \geq 0$, let $K_i' \subseteq X'$ be the subset consisting of those sets whose ith coordinate is zero, and choose a representative $K_i \in \Sigma$ for $\phi([K_i']) \in \Sigma/\Sigma_0$. Now define $f : X \to 2^\omega$ by the formula

$$f(x)_i = \begin{cases} 0 & \text{if } x \in K_i \\ 1 & \text{otherwise.} \end{cases}.$$

By construction, we have $[f^{-1}K_i] = \phi([K_i])$ for each i. Since the sets K_i generate the σ-algebra of Borel subsets of X', we conclude that $[f^{-1}Y] = \phi([Y])$ for all Borel sets Y. Taking Y to be a set of measure zero, we deduce that f is quasi-measure preserving; it is now clear that f has the desired properties.

□

Having decided that the class of standard measure spaces is a good class to look at, we can now rephrase our basic question as follows: which von Neumann algebras A have the form $L^\infty(X)$, where X is standard?

Theorem 9. *Let A be an abelian von Neumann algebra. The following conditions are equivalent:*

(a) *A is separable (in the sense of the last lecture).*

(b) *The von Neumann algebra A has the form $L^\infty(X)$, where (X, Σ, μ) is a standard finite measure space.*

Corollary 10. *The construction $(X, \Sigma, \mu) \mapsto L^\infty(X)$ determines an equivalence from the category $\mathcal{M}_s$ of standard measure spaces (and quasi-measure preserving maps) to the category of separable abelian von Neumann algebras.*

Proof of Proposition 9. The implication $(b) \Rightarrow (a)$ is easy: if $A = L^\infty(X)$ for X standard, then A has a faithful representation on the separable Hilbert space $L^2(X)$. Assume now that (a) is satisfied. Then A does not admit any uncountable families of mutually orthogonal projections, so we can write A as a countable product of von Neumann algebras of the form $L^\infty(X)$, where X is a finite measure space. Taking the union of these measure spaces (and scaling the measures appropriately), we may assume that $A = L^\infty(X)$ for some finite measure space X. Choose a countable subset S of A which is ultraweakly dense. Since step functions are dense in A with respect to the norm topology, we may assume that each element of S is a finite linear combination of projections corresponding to measurable subsets of X. Let S_0 be the (countable) collection of all measurable subsets which arise in this way. It is easy to see that S_0 generates the Σ-algebra Σ/Σ_0. Thus, (a) implies the following:

(a') A is isomorphic to $L^\infty(X)$ for some finite measure space (X, Σ, μ) for which the σ-algebra Σ/Σ_0 is countably generated.

We will complete the proof by showing that $(a') \Rightarrow (b)$. Breaking X up into its atomic and nonatomic parts, we can reduce to two special cases:

(*i*) X is union of atoms $K_1 \cup K_2 \cup \cdots$. Then $L^\infty(X)$ is isomorphic to $L^\infty(\{x_1, x_2, \ldots\})$.

(*ii*) X is atomless. If $\mu(X) = 0$ there is nothing to prove; otherwise we may assume (after rescaling) that $\mu(X) = 1$. Choose a countable sequence of measurable subsets $E_1, E_2, \ldots$ which generate Σ/Σ_0 as a Σ-algebra. We will construct a measure-preserving function $f : X \to [0, 1]$ such that each E_i is equal (modulo sets of measure zero) to $f^{-1}K$, for some Borel set $K \subseteq [0, 1]$. This will be sufficient: if we let Σ' denote the σ-algebra of Borel sets of $[0, 1]$, then (since f is measure preserving) we get an induced map $\rho : \Sigma'/\Sigma'_0 \to \Sigma/\Sigma_0$. This map is injective (again since f is measure preserving) and also surjective, since Σ/Σ_0 is generated by the sets E_i, which belong to the image of ρ.

It remains to construct f. This will have to wait for the next lecture.

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