# Math 261y: von Neumann Algebras (Lecture 13)

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In this lecture, we will begin the study of *abelian* von Neumann algebras. We first describe the prototypical example of an abelian von Neumann algebra. Let $(X, \mu)$ be a measure space: that is, $X$ is a set equipped with a $\sigma$-algebra of subsets of $X$, called *measurable set*, and $\mu$ is a countably additive measure defined on this $\sigma$-algebra. Let us assume furthermore that $\mu$ is finite (that is, $\mu(X) < \infty$). We let $L^\infty(X)$ denote the collection of measurable functions on $X$ which are essentially bounded, modulo sets supported on a subset of $X$ having measure 0. We let $L^2(X)$ denote the Hilbert space of square-integrable functions on $X$, again defined modulo functions supported on a subset of measure 0. The assumption that $\mu$ is finite guarantees that we have an inclusion

$$L^\infty(X) \subset L^2(X).$$

We regard $L^\infty(X)$ as an algebra, with the algebra structure given by pointwise multiplication. In fact, it is a $C^*$-algebra, with conjugation given by

$$f^*(x) = \overline{f(x)}$$

and the essential supremum norm. This $C^*$-algebra acts on the Hilbert space $L^2(X)$: that is, we have a $*$-algebra homomorphism $L^\infty(X) \to B(L^2(X))$. This map is injective: note that the inclusion of $L^\infty(X)$ into $L^2(X)$ is just given by the action of $L^\infty(X)$ on the constant function $\underline{1} \in L^2(X)$.

**Theorem 1.** *Let $(X, \mu)$ be a finite measure space. Then $L^\infty(X)$ is a von Neumann algebra (when regarded as a subalgebra of $B(L^2(X))$).*

*Proof.* We will show that $L^\infty(X)$ is equal to its own commutant (and therefore also equal to its own double commutant). Let $T \in B(L^2(X))$ be a bounded operator and let $f = T(\underline{1}) \in L^2(X)$. We will prove that the function $f$ belongs to $L^\infty$ and $T$ is given by multiplication by $f$. Let $C \in R_{\geq 0}$ be the norm of the operator $T$. We may assume $C > 0$ (otherwise $T = 0$ and the result is obvious). We claim that the essential supremum of $|f|$ is $\leq C$. Assume otherwise; then there exists a measurable set $Y \subseteq X$ of positive measure such that $|f| > C$ on the subset $Y$. Define a function $g : X \to \mathbf{C}$ by the formula

$$g(x) = \begin{cases} \frac{1}{f(y)} & \text{if } y \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in L^\infty$, so we have

$$T(g) = (Tg)(\underline{1}) = gT(\underline{1}) = gf.$$

Note that $gf$ is the characteristic function of $Y$, and therefore has $L^2$-norm $\mu(Y)$. We get

$$\mu(Y) = ||gf||^2 = ||T(g)||^2 \leq C^2 ||g||^2$$

Since $g$ is supported on $Y$ and has absolute value $< \frac{1}{C}$ on $Y$, we get $||g||^2 < \frac{\mu(Y)}{C^2}$. Combining these, we get $\mu(Y) < \mu(Y)$, a contradiction. This completes the proof that $f \in L^\infty(X)$. Moreover, the above argument shows that $T(g) = gf$ for all $g \in L^\infty(X)$. Since $L^\infty$ is dense in $L^2$ (and both $T$ and multiplication by $f$ are continuous functions), we conclude that $T(g) = gf$ for all $g \in L^2(X)$. □

Suppose now that $A$ is a commutative $C^*$-algebra, and that $V$ is a cyclic representation of $A$. Choosing a generating unit vector $v \in V$, we can write $V = V_\rho$ where $\rho : A \to \mathbf{C}$ is the state given by $\rho(x) = (x(v), v)$. Let $X = \operatorname{Spec} A$. Then $\rho$ can be regarded as a positive, continuous functional on $C^0(X)$, which we can identify with a finite Baire measure on $X$. This measure is characterized by the formula

$$\rho(f) = \int f d\mu$$

for every continuous function $f$ on $X$. In particular, we can identify $V = V_\rho$ with the completion of $C^0(X)$ with respect to the inner product

$$\langle f, g \rangle = \rho(g^* f) = \int \overline{g} f d\mu.$$

This inner product is given by restricting the natural inner product on $L^2(X, \mu)$. Since $C^0(X)$ is dense in $L^2(X, \mu)$, we obtain an isomorphism of Hilbert spaces $V \simeq L^2(X, \mu)$.

**Proposition 2.** *In the above situation, the image of $C^0(X)$ in $B(L^2(X))$ generates $L^\infty(X)$ as a von Neumann algebra: that is, $L^\infty(X)$ is the double commutant of $C^0(X)$.*

*Proof.* Let $A$ denote the von Neumann algebra generated by $C^0(X)$ (in $B(L^2(X))$). Since $L^\infty(X)$ is a von Neumann algebra containing the image of $C^0(X)$, we clearly have $A \subseteq L^\infty(X)$. We wish to prove the converse inclusion. Let $\mathcal{B}$ denote the collection of all Baire subsets of $X$: that is, the $\sigma$-algebra generated by sets of the form $f^{-1} \mathbb{R}_{\geq 0}$, where $f : X \to \mathbb{R}$ is a continuous function. For each $K \in \mathcal{B}$, let $\chi_K$ denote the characteristic function of $K$. The collection of all finite linear combinations of characteristic functions is dense in $L^\infty(X)$ with respect to the norm topology. It will therefore suffice to show that each $\chi_K \in A$. Let $\mathcal{B}_0$ denote the subset of $\mathcal{B}$ consisting of those Baire sets $K$ such that $\chi_K \in A$; we wish to show that $\mathcal{B}_0 = \mathcal{B}$. Note that $\chi_{X-K} = 1 - \chi_K$, so that $\mathcal{B}_0$ is closed under the formation of complements. Since $\chi_{K \cap K'} = \chi_K \chi_{K'}$, the set $\mathcal{B}_0$ is closed under the formation of finite intersections. Passing to complements, we see that it is also closed under finite unions. We claim that it is also closed under the formation of countable unions: that is, if $K_1, K_2, \ldots \in \mathcal{B}_0$, then the intersection $K = \bigcup K_i$ also belongs to $\mathcal{B}_0$. Replacing $K_n$ by $\bigcup_{m \leq n} K_m$, we can assume that the sequence of sets $K_n$ is increasing. We claim that in this case, the sequence of functions $\chi_{K_i}$ converges to $\chi_K$ in the weak topology. To prove this, it suffices to show that for every pair of square-integrable functions $f$ and $g$, the sequence of integrals

$$\int \overline{g} f \chi_{K_n} d\mu$$

converges to $\int \overline{g} f \chi_K d\mu$, which follows from the integrability of $\overline{g} f$.

We have shown that $\mathcal{B}_0$ is a $\sigma$-algebra of subsets of $X$. To complete the proof that $\mathcal{B}_0 = \mathcal{B}$, it will suffice to show that for every continuous function $f : X \to \mathbb{R}$, the inverse image $U = f^{-1} \mathbb{R}_{\geq 0}$ belongs to $\mathcal{B}_0$. To prove this, we note that the characteristic function $\chi_U$ is a weak limit of the sequence of functions $h_n$ given by

$$h_n(x) = \begin{cases} 0 & \text{if } f(x) \leq 0 \\ n f(x) & \text{if } 0 \leq f(x) \leq \frac{1}{n} \\ 1 & \text{if } f(x) = \frac{1}{n}. \end{cases}$$

□

We will need the following simple fact.

**Proposition 3.** *Let $\phi : A \to B$ be an ultraweakly continuous $*$-algebra homomorphism between von Neumann algebras. Then the image of $\phi$ is a von Neumann algebra: that is, it is closed with respect to the ultraweak topology on $B$.*

*Proof.* Replacing $A$ by $A/\ker(\phi)$, we may suppose that $\phi$ is injective, and therefore an isometry. Because of the Krein-Smulian theorem proved in the last lecture, it will suffice to show that $\mathrm{Im}(\phi) \cap B_{\leq 1}$ is ultraweakly closed in $B$. Since $\phi$ is an isometry, this is given by the image of $A_{\leq 1}$ under the map $\phi$. Since $A_{\leq 1}$ is compact for the ultraweak topology, its image is also compact, and therefore closed. □

We can now show that essentially all abelian von Neumann algebras are of the type described in Theorem 1.

**Theorem 4.** *Let $A$ be an abelian von Neumann algebra. Then there exists a collection of mutually orthogonal projections $\{e_\alpha\}$ such that $\sum e_\alpha = 1$ and each of the von Neumann algebras $Ae_\alpha$ is isomorphic to $L^\infty(X_\alpha)$, where $X_\alpha$ is a compact Hausdorff space equipped with a finite Baire measure.*

*Proof.* Let $\{e_\alpha\}$ be a maximal collection of mutually orthogonal projections with the desired property; we wish to show that $e = \sum e_\alpha$ is the identity. Assume otherwise. Let $A' = (1-e)A$, and identify $A'$ with a von Neumann algebra in $B(V)$ for some Hilbert space $V$. Choose a nonzero vector $v \in V$, and let $V_0 = \overline{A'v}$. Then $V_0$ is an ultraweakly continuous representation of $A'$ equipped with a cyclic vector. It follows that $V_0 \simeq L^2(X)$ for some finite Baire measure on the space $X = \mathrm{Spec}\, A'$. The map $A' \simeq C^0(X) \rightarrow B(V_0)$ is ultraweakly continuous, so its image is a von Neumann subalgebra of $B(V_0)$ by Proposition 3. Using Proposition 2, we see that this image is precisely $L^\infty(X)$. We therefore obtain an isomorphism of von Neumann algebras $A' \simeq L^\infty(X) \times A''$, which contradicts the maximality of the collection $\{e_\alpha\}$. □

Suppose that $A \subseteq B(V)$ is a von Neumann algebra equipped with a collection of mutually orthogonal central projections $\{e_\alpha\}$ with $\sum e_\alpha = 1$. Then $V$ decomposes as an orthogonal direct sum $\bigoplus e_\alpha(V)$, and each $Ae_\alpha$ can be identified with a von Neumann algebra in $B(e_\alpha(V))$. In this case, we can recover $A$ uniquely as the subalgebra of the product

$$\prod_\alpha Ae_\alpha$$

consisting of those elements $(x_\alpha)$ such that the set $\{||x_\alpha||\}$ is bounded. It is clear that every such element determines a bounded operator on $V \simeq \bigoplus e_\alpha(V)$, which is a strong limit of the operators given by the sums $\sum_{\alpha \in S} x_\alpha$ where $S$ ranges over finite collections of indices.

In particular, if each $Ae_\alpha$ has the form $L^\infty(X_\alpha)$ for some measure space $X_\alpha$, we conclude that $A$ is isomorphic to $L^\infty(X)$, where $X$ is the disjoint union of the measure spaces $X_\alpha$. Invoking Theorem 4, we deduce:

**Theorem 5.** *Let $A$ be an abelian von Neumann algebra. Then there exists a measure space $X$ and an isomorphism $A \simeq L^\infty(X)$.*

Note that Theorem 5 is not quite a converse to Theorem 1, because we do not assert that the measure on $X$ is finite. This can usually be arranged in practice. For example, if $A \subseteq B(V)$ for a *separable* Hilbert space $V$, then the decomposition of Theorem 4 will be countable; then the measure space $X$ constructed in the above discussion will be $\sigma$-finite, hence equivalent (after rescaling the measure) to a finite measure.