Math 261y: von Neumann Algebras (Lecture 12)

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In this lecture, we will complete our algebraic characterization of von Neumann algebra morphisms by proving the following result:

Lemma 1. Let A be a von Neumann algebra and let $\mu : A \to \mathbb{C}$ be a linear functional. If μ is ultra-strongly continuous on the unit ball $A_{\leq 1}$, then μ is ultraweakly continuous.

In fact, we will prove the following:

Proposition 2. Let A be a von Neumann algebra with unit ball $A_{\leq 1}$, and let $\mu : A \to \mathbf{C}$ be a linear functional. The following conditions are equivalent:

- (a) μ is ultraweakly continuous.
- (a') The kernel ker(μ) is closed in the ultraweak topology.
- (b) μ is ultrastrongly continuous.
- (b') The kernel ker(μ) is closed in the ultrastrong topology.
- (c) μ is ultraweakly continuous on $A_{\leq 1}$.
- (c') The set $\ker(\mu) \cap A_{\leq 1}$ is ultraweakly closed.
- (d) μ is ultrastrongly continuous on $A_{<1}$.
- (d') The set $\ker(\mu) \cap A_{\leq 1}$ is ultrastrongly closed.

We have an obvious web of implications



In particular, condition (a) is the strongest and condition (d') is the weakest. The results of the last lecture shows that a completely additive state satisfies (d), and that a state which satisfies (a) is completely additive. We will prove Proposition 2 by showing that $(d') \Rightarrow (a)$. Actually, we will proceed by showing that

$$(d') \Rightarrow (c') \rightarrow (a') \Rightarrow (a).$$

The implications $(a') \Rightarrow (a)$ and $(b') \Rightarrow (b)$ are easy. If ker (μ) is closed (for whatever topology), then the quotient topology on $A/\ker(\mu)$ is Hausdorff, and therefore agrees with the standard topology on $A/\ker(\mu) \simeq \mathbf{C}$. It follows that the composite map $A \to A/\ker(\mu) \to \mathbf{C}$ is continuous.

Lemma 3. We have $(b) \Rightarrow (a)$. That is, every ultrastrongly continuous functional on a von Neumann algebra $A \subseteq B(V)$ is ultraweakly continuous.

Proof. Let $\mu : A \to B(V)$ be ultrastrongly continuous. Then there exists a vector $v \in V^{\oplus \infty}$ such that $|\mu(x)| \leq ||x(v)||$ for each $x \in A$. Replacing V by V^{∞} , we can assume $v \in V$. Define a functional $\mu_0 : Av \to \mathbb{C}$ by the formula $\mu_0(x(v)) = \mu(x)$ (this is well-defined: if x(v) = y(v), then (x-y)(v) = 0, so that $\mu(x-y) = 0$ and $\mu(x) = \mu(y)$). We have $|\mu_0(x(v))| = |\mu(x)| \leq ||x(v)||$, so that μ_0 has operator norm ≤ 1 . It follows that μ_0 extends to a continuous functional on the closure $V_0 = \overline{Av} \subseteq V$. Since V_0 is a Hilbert space, this functional is given by inner product with some vector $w \in V_0$. Then

$$\mu(x) = (x(v), w),$$

so that μ is ultraweakly continuous.

We will need the following basic result from the theory of convexity:

Theorem 4. Let W be a locally convex topological vector space (over the real numbers, say), and let $K \subseteq W$. The following conditions are equivalent:

- (1) The set K is closed and convex.
- (2) There exists a collection of continuous functionals $\lambda_{\alpha} : W \to \mathbb{R}$ and a collection of real numbers C_{α} such that $K = \{w \in W : (\forall \alpha) [\lambda_{\alpha}(w) \ge C_{\alpha}]\}.$

Proof. We may assume without loss of generality that K contains the origin. Let $v \in W - K$. Since K is closed, there exists an open neighborhood U of the origin such that $(v + U) \cap K = \emptyset$. Since W is locally convex, we can assume that U is convex. Then K + U is a convex subset of the origin. For $w \in W$, define

$$||w|| = \inf\{t \in \mathbb{R}_{>0} : tw \in K + U\}.$$

This is *almost* a prenorm on W: the convexity of K + U gives

$$||w + w'|| \le ||w|| + ||w'||,$$

and we obviously have

$$||tw|| = t||w||$$

for $t \ge 0$. This generally does not hold for t < 0: that is, we can have $||w|| \ne ||-w||$. Note that $||v|| \ge 1$ (since $v \notin K + U$). Define $\mu : \mathbb{R} v \to \mathbb{R}$ by the formula $\mu(tv) = t$, so that μ satisfies the inequality $\mu(w) \le ||w||$ for $w \in \mathbb{R} v$. The proof of the Hahn-Banach theorem allows us to extend μ to a function on all of W satisfying the same condition. We have $|\mu(w)| = \pm \mu(w) \le 1$ for $w \in U \cap -U$, so that μ is continuous. Since $\mu(v) = 1$, μ does not vanish so there exists $u \in U$ with $\mu(u) = \epsilon > 0$. Then for $k \in K$, we have $\mu(k+u) \le 1$, so that $\mu(k) \le 1 - \mu(u)$. Then

$$\{w \in W : \mu(w) \le 1 - \mu(u)\}$$

is a closed half-space containing K which does not contain v.

Corollary 5. Let A be a von Neumann algebra, and let $K \subseteq A$ be a convex subset. Then K is closed for the ultraweak topology if and only if K is closed for the ultrastrong topology.

From Corollary 5 we get the implications $(b') \Rightarrow (a')$ and $(d') \Rightarrow (c')$. To complete the proof, it suffices to show that $(c') \Rightarrow (a')$. Recall that A admits a Banach space predual E, and that the ultraweak topology on A coincides with the weak *-topology. The implication $(c') \Rightarrow (a')$ is a special case of the following more general assertion:

Theorem 6 (Krein-Smulian). Let E be a real Banach space and let $K \subseteq E^{\vee}$ be a convex set. For each real number $r \geq 0$, we let $E_{\leq r}^{\vee}$ denote the closed unit ball of radius r in E^{\vee} . If each of the intersections $K_{\leq r} = K \cap E_{\leq r}^{\vee}$ is closed for the weak *-topology, then K is closed for the weak *-topology.

Proof. Let $v \in E - K$; we wish to show that v does not belong to the weak *-closure of K. Replacing K by K - v, we can reduce to the case where v = 0. Since each $K_{< r}$ is closed in the weak *-topology, it is also closed in the norm topology. It follows that K is closed in the norm topology. In particular, since $0 \notin K$, there exists a real number $\epsilon > 0$ such that $E_{<\epsilon}^{\lor}$ does not intersect K. By rescaling, we can assume that $\epsilon = 1$. We construct a sequence of finite subsets $S_1, S_2, S_3, \ldots \subseteq E$ with the following properties:

- (a) If $\mu \in K_{\leq n+1}$, then there exists $v \in S_1 \cup \cdots \cup S_n$ such that $\mu(v) > 1$.
- (b) If $v \in S_n$, then $||v|| = \frac{1}{n}$.

Assume that S_1, \ldots, S_{n-1} have been constructed, and set

$$K(n) = K_{\leq n+1} \cap \{ \mu \in E : (\forall v \in S_1 \cup \dots \cup S_{n-1}) [\mu(v) \leq 1] \}$$

Then K(n) is a weak *-closed subset of E^{\vee} which is bounded in the norm topology, and is therefore weak *-compact. By construction, K(n) does not intersect $K_{\leq n}$. It follows that if $\mu \in K(n)$, then $||\mu|| > n$. We may therefore choose a vector $v \in E$ with $||v|| = \frac{1}{n}$ such that μ belongs to the set $U_v = \{\rho \in E^{\vee} : \rho(v) > 1\}$ (which is open for the weak *-topology). Since K(n) is compact, we can cover K(n) by finitely many such open sets $U_{v_1}, U_{v_2}, ..., U_{v_m}$. We then take $S_n = \{v_1, ..., v_m\}$.

Let $S = \bigcup S_i$. Then S is a countable subset of E; we can enumerate its elements as v_1, v_2, \ldots (if S is finite, we can extend this sequence by adding a sequence of zeros at the end). By construction, this sequence converges to zero in the norm topology on E. Let $C^0(\mathbf{Z}_{>0})$ denote the Banach space consisting of continuous maps

$$\mathbf{Z}_{>0} \to \mathbb{R}$$

which vanish at infinity: that is, the Banach space of sequences $(\lambda_1, \lambda_2, \ldots)$ which converge to zero. We have a map

$$f: E^{\vee} \to C^0(\mathbf{Z}_{>0})$$

given by $\mu \mapsto (\mu(v_1), \mu(v_2), \cdots)$. The image f(K) is a convex subset of $C^0(\mathbf{Z}_{>0})$. By construction, if $\mu \in K$ then $\mu(v_i) > 1$ for some i, so that f(K) does not intersect the unit ball of $C^0(\mathbf{Z}_{>0})$. It follows that 0 does not belong to the closure of f(K). Applying Theorem 4, we see that there is a continuous functional $\rho: C^0(\mathbf{Z}_{>0}) \to \mathbb{R}$ such that $\rho(K) \subseteq \mathbb{R}_{>1}$. This functional is given by a summable sequence of real numbers (c_1, c_2, \ldots) , and satisfies

$$\sum c_i \mu(v_i) \ge 1$$

for each $\mu \in K$. Set $v = \sum c_i v_i$; then $v \in E$ is a vector satisfying $\mu(v) \ge 1$ for $\mu \in K$. Then K is contained in the weak *-closed set $\{\mu \in E^{\vee} : \mu(v) \ge 1\}$, so that 0 does not belong to the closure of K.