Math 261y: von Neumann Algebras (Lecture 11)

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In the last lecture, we promised a proof of the following assertion:

Theorem 1. *Let $\phi : A \to B$ be a $*$-algebra homomorphism between von Neumann algebras. If ϕ is completely additive, then it is ultraweakly continuous.*

Our goal in this lecture is to prove Theorem 1. Assume that $\phi : A \to B$ is completely additive; we wish to show that it is ultraweakly continuous. Using the definition of the ultraweak topology on B, this is equivalent to the following assertion:

(a) For every ultraweakly continuous linear functional $\mu : B \to \mathbf{C}$, the composite map $\mu \circ \phi : A \to \mathbf{C}$ is an ultraweakly continuous functional on A.

The next step is to observe that it suffices to consider ultraweakly continuous *states* of B, by virtue of the following:

Lemma 2. *Let $B \subseteq B(V)$ be a von Neumann algebra. Then the vector space of ultraweakly continuous functionals on B is generated by ultraweakly continuous states.*

Proof. Every ultraweakly continuous functional $\mu : B \to \mathbf{C}$ is given by

$$\mu(x) = \sum (x(v_i), w_i)$$

for some sequences of vectors $v_i, w_i \in V$ with $\sum ||v_i||^2 < \infty, \sum ||w_i||^2 < \infty$. Replacing V by $V^{\oplus \infty}$, we may assume that μ is given by $\mu(x) = (x(v), w)$. Then

$$\mu(x) = \frac{1}{4}(x(v+w), v+w) + \frac{i}{4}(x(v+iw), v+iw) + -\frac{1}{4}(x(v-w), v-w) - \frac{i}{4}(x(v-iw), v-iw)$$

is a linear combination of ultraweakly continuous positive functionals, each of which is a multiple of an ultraweakly continuous state. □

Returning to the proof of Theorem 1, we are reduced to proving the following:

(b) Let $\phi : A \to B$ be completely additive, and let $\mu : B \to \mathbf{C}$ be an ultraweakly continuous state. Then $\mu \circ \phi$ is an ultraweakly continuous state on A.

Definition 3. Let A be a von Neumann algebra and let $\mu : A \to \mathbf{C}$ be a state. We will say that μ is *completely additive* if, for every collection $\{e_\alpha\}$ of mutually orthogonal projections on A, we have $\mu(\sum e_\alpha) = \sum \mu(e_\alpha)$.

It is clear that every ultraweakly continuous state is completely additive. Moreover, if $\phi : A \to B$ is a completely additive $*$-algebra homomorphism and $\mu : B \to \mathbf{C}$ is a completely additive state, then $\mu \circ \phi$ is a completely additive state on A. We are therefore reduced to proving the following:

Proposition 4. *Let $A \subseteq B(V)$ be a von Neumann algebra and let $\mu : A \to \mathbf{C}$ be a completely additive state. Then μ is ultraweakly continuous.*

We will break the proof of Proposition 4 into two parts:

Lemma 5. *Let $A \subseteq B(V)$ be a von Neumann algebra and let $\mu : A \to \mathbf{C}$ be a completely additive state. Then μ is ultra-strongly continuous when restricted to the unit ball $A_{\leq 1}$.*

Lemma 6. *Let A be a von Neumann algebra and let $\mu : A \to \mathbf{C}$ be a linear functional. If μ is ultra-strongly continuous on the unit ball $A_{\leq 1}$, then μ is ultraweakly continuous.*

The proof of Lemma 6 uses some ideas from functional analysis and will be given in the next lecture. Let us concentrate on Lemma 5. We will need a short digression concerning the structure of abelian von Neumann algebras.

Proposition 7. *Let $A \subseteq B(V)$ be a commutative von Neumann algebra, and set $X = \operatorname{Spec} A$ (so that $A \simeq C^0(X)$). Then the compact Hausdorff space X has the following property: for every continuous function $f : X \to [-1, 1]$, there exists a decomposition $X = X_- \coprod X_+$ into clopen subsets where $\{x \in X : f(x) < 0\} \subseteq X_-$ and $\{x \in X : f(x) > 0\} \subseteq X_+$.*

Proof. Define $f_+ : X \to [0, 1]$ by the formula

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_+ \in A$ determines a self-adjoint operator on V, and therefore determines an orthogonal decomposition $V \simeq \ker(f_+) \oplus \overline{f_+(V)}$. Since f_+ belongs the center of A, this decomposition is A-invariant. Let e denote the projection onto $\ker(f_+)$, so that $e \in A'' = A$ can be identified with a function on X. Since e is idempotent, this determines a decomposition of X into pieces $X_- = \{x \in X : e(x) = 0\}$ and $X_+ = \{x \in X : e(x) = 1\}$. We claim that X_- and X_+ have the desired properties. To prove this, we note that e is the identity on $f_+(V)$, so that $ef_+ = f_+$ and therefore X_+ contains the support of the function f_+. Similarly, if we set $f_- = f - f_+$, then $ef_- = 0$ so that X_- contains the support of f_-. □

Remark 8. The property of X appearing in the statement of Proposition 7 should seem very strange. It is not satisfied for any metric space (except those equipped with the discrete topology).

Corollary 9. *Let $A \subseteq B(V)$ be a commutative von Neumann algebra. Then every nonnegative function on $X = \operatorname{Spec} A$ can be written as a limit of nonnegative locally constant functions.*

Proof. Let $f : X \to [0, 1]$ be a continuous function. Using Proposition 7, we can decompose X into clopen pieces X_- and X_+ containing $f^{-1}[0, \frac{1}{2})$ and $f^{-1}(\frac{1}{2}, 1]$. Set

$$f_0(x) = \begin{cases} \frac{1}{2} & \text{if } x \in X_+ \\ 0 & \text{if } x \in X_-. \end{cases}$$

Then $f - f_0$ is a continuous function with values in $[0, \frac{1}{2}]$. Iterating this procedure, we get arbitrarily good approximations to f. □

Corollary 10. *Let $A \subseteq B(V)$ be a von Neumann algebra and let $x \in A$ be a positive element. Then x can be written as a limit (in the norm topology) of positive linear combinations of mutually orthogonal projections.*

Proof. Replacing A by the ultraweak closure of $\mathbf{C}[x]$, we can reduce to the case where A is commutative, in which case the desired result follows from Corollary 9. □

Let A be a von Neumann algebra, and let $\mu, \mu' : A \to \mathbf{C}$ be norm-continuous linear functionals. We write $\mu \leq \mu'$ if the difference $\mu' - \mu$ is a positive functional: that is, if $\mu(x) \leq \mu'(x)$ for every positive element $x \in A$. Using Corollary 10, we see that it suffices to check that $\mu(e) \leq \mu'(e)$ for every projection $e \in A$.

Proposition 11. *Let $A \subseteq B(V)$ be a von Neumann algebra. Let $\mu : A \to \mathbf{C}$ be a positive linear functional, let $v \in V$ be a vector with $||v||^2 = \mu(1)$, and let $\mu_v : A \to \mathbf{C}$ be given by $\mu_v(x) = (x(v), v)$. Then there exists a nonzero projection $e \in A$ such that $\mu \leq \mu_v$ when restricted to the von Neumann algebra eAe.*

Proof. Let S be the set of projections e' in A such that $\mu_v(e') < \mu(e')$, and let T be the collection of all subsets of S consisting of mutually orthogonal elements. Using Zorn's lemma, we deduce that T has a maximal element $S_0 \subseteq S$. If S_0 is empty, then $\mu(e') \leq \mu_v(e')$ for every projection e', so that $\mu \leq \mu_v$ and we are done.

Otherwise, let $f = \sum_{e' \in S_0} e'$. We have

$$\mu_v(f) = \sum_{e' \in S_0} \mu_v(e') < \sum_{e' \in S_0} \mu(e') \leq \mu(f) \leq \mu(1).$$

Since $\mu_v(1) = ||v||^2 = \mu(1)$, we must have $f \neq 1$. Set $e = 1 - f$. Since every projection in eAe is orthogonal to each e_i, we conclude from the maximality of S that $\mu \leq \mu_v$ on eAe. □

Now let $\mu : A \to \mathbf{C}$ be a state. Using Zorn's lemma, we can choose a maximal collection of pairs (e_α, v_α) such that the e_α are mutually orthogonal projections belonging to A, and $\mu \leq \mu_{v_\alpha}$ when restricted to $e_\alpha A e_\alpha$. Set $e = \sum e_\alpha$ and $e' = 1 - e$. If $e \neq 1$, then we can apply Proposition 11 to the von Neumann algebra $e'Ae'$ (acting on $e'V \neq 0$) to contradict the maximality of our collection.

We now prove Lemma 5 by showing that if μ is completely additive, then the restriction of μ to $A_{\leq 1}$ is ultra-strongly continuous. For this, we must show that for each $\epsilon > 0$, there exists an ultra-strongly open neighborhood $U \subseteq A$ of 0, such that $|\mu(x)| \leq \epsilon$ for $x \in U \cap A_{\leq 1}$. Since μ is completely additive, we have

$$1 = \mu(1) = \sum_\alpha \mu(e_\alpha).$$

We may therefore choose a finite subset $\{e_1, \ldots, e_n\}$ of our projections such that

$$\mu(e_1) + \cdots + \mu(e_n) \geq 1 - \frac{\epsilon^2}{4}$$

. Let $q = 1 - e_1 - \cdots - e_n$, so that $\mu(q) \leq \frac{\epsilon^2}{4}$. For each $x \in A_{\leq 1}$, we have

$$|\mu(x)| \leq |\mu(xq)| + \sum_{1 \leq i \leq n} |\mu(xe_i)|$$

Since the pairing $(a, b) \mapsto \mu(b^* a)$ is positive semidefinite on A, the Cauchy-Schwartz inequality gives $|\mu(b^* a)| \leq |\mu(b^* b)|^{\frac{1}{2}} |\mu(a^* a)|^{\frac{1}{2}}$. In particular we get

$$|\mu(xq)| \leq |\mu(x^* x)|^{\frac{1}{2}} |\mu(q)|^{\frac{1}{2}}$$

Since $x^* x$ is a positive element of the unit ball $A_{\leq 1}$, we have $x^* x \leq 1$, so that $|\mu(x^* x)| \leq 1$. By construction, we have $|\mu(q)| \leq \frac{\epsilon^2}{4}$. Combining these facts, we get

$$|\mu(xq)| \leq \frac{\epsilon}{2}.$$

Applying the Cauchy-Schwartz inequality again gives

$$|\mu(xe_i)| = |\mu(1(xe_i))| \leq |\mu(1)|^{\frac{1}{2}} |\mu(e_i x^* x e_i)|^{\frac{1}{2}} \leq ||xe_i v_i||^{\frac{1}{2}}.$$

We therefore get

$$|\mu(x)| \leq \frac{\epsilon}{2} + \sum_{1 \leq i \leq n} ||x(e_i v_i)||^{\frac{1}{2}}.$$

This is $\leq \epsilon$ on the intersection

$$A_{\leq 1} \cap \{x \in A : (\forall 1 \leq i \leq n)[||x(e_i v_i)|| < \frac{\epsilon}{2n}]\}$$

where the latter set is an open neighborhood of 0 in the strong topology.