Math 261y: von Neumann Algebras (Lecture 1)

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Let G be a compact group. The representation theory of G is completely reducible: every finite-dimensional representation can be written as a direct sum of irreducible representations. Suppose we want to consider also infinite-dimensional representations V. For this to be sensible, we should assume that V is equipped with some sort of topology. Let us restrict our attention to the easiest case: assume that V is a (complex) Hilbert space and that the representation of G on V is unitary. In this case, we also have complete reducibility: V can be written as a (generally infinite) direct sum of irreducible representations of G, each of which is finite dimensional.

A basic problem in representation theory is to obtain a generalization of the above picture to the case where G is a locally compact group, not necessarily compact. In this case, the situation is much more complicated. Suppose, for example, that $G = \mathbb{R}$ is the group of real numbers (under addition). The irreducible (unitary) representations of G are all one-dimensional, given by characters

$$\mathbb{R} \to \mathbf{C}^*$$

$$t \mapsto e^{2\pi i \lambda t}$$

for $\lambda \in \mathbb{R}$. However, not every unitary representation of G can be written as a direct sum of irreducible representations. For example, the regular representation $L^2(\mathbb{R})$ of G does not contain any irreducible summands at all.

To fix ideas, let us take V to be a Hilbert space equipped with a unitary action of a group G (for the time being, we can ignore any topology on G). We can then try to study all ways of breaking V up into "smaller" pieces. Note that for any closed G-invariant subspace $W \subseteq V$, we obtain a G-invariant direct sum decomposition $V \simeq W \oplus W^{\perp}$. This direct decomposition is given by $(e_W, 1 - e_W)$, where $e_W : V \to W$ denotes the orthogonal projection onto the subspace W, and $1 - e_W = e_{W^{\perp}}$ the orthogonal projection onto $W^{\perp}$. Then e_W is a bounded operator on the Hilbert space V, which commutes with the action of G. Moreover, e_W is a *projection*: that is, it is a self-adjoint operator satisfying $e_W^2 = e_W$. This analysis has a converse: if $e : V \to V$ is any G-invariant projection, then the image eV is a closed, G-invariant subspace of V (with complement $(1-e)V$). We can therefore reformulate our problem as follows:

($*$) Determine all G-invariant projections $e : V \to V$.

Notation 1. If V is a Hilbert space, we let $B(V)$ denote the algebra of all bounded operators on V. For $f \in B(V)$, we let f^* denote the adjoint of A (characterized by the formula $(v, f^* w) = (fv, w)$). Given a subset $S \subseteq B(V)$, we let S' denote the set $\{f \in B(V) : (\forall s \in S) fs = sf\}$. We refer to S' as the *commutant* of S. It is a subalgebra of $B(V)$. Moreover, if S is stable under the operation $f \mapsto f^*$, then S' is a $*$-subalgebra: that is, it is also stable under the operation $f \mapsto f^*$.

We can therefore rephrase our problem once again:

($*'$) Let G act on a Hilbert space V via a unitary representation $\alpha : G \to B(V)$. Determine all projections which belong to the commutant $\alpha(G)'$.

Since α is a unitary representation, the image $\alpha(G)$ is closed under $*$ (we have $\alpha(g)^* = \alpha(g^{-1})$), so that $\alpha(G)'$ is a $*$-subalgebra of $B(V)$.

Theorem 2 (von Neumann Double Commutant Theorem)**.** *Let V be a Hilbert space and let $A \subseteq B(V)$ be a $*$-subalgebra. The following conditions are equivalent:*

(1) *There exists a subset $S \subseteq B(V)$ such that $A = S'$.*

(2) *We have $A = A''$.*

(3) *The algebra A is closed in the strong topology on $B(V)$.*

(4) *The algebra A is closed in the weak topology on $B(V)$.*

(5) *The algebra A is closed in the ultrastrong topology on $B(V)$.*

(6) *The algebra A is closed in the ultraweak topology on $B(V)$.*

The equivalence of (1) and (2) is easy. It is trivial that (2) $\rightarrow$ (1) (take $S = A'$). Conversely, suppose (1) is satisfied. We always have $A \subseteq A''$. If $A = S'$, then $S \subseteq A'$ so that $A'' \subseteq S' = A$. The equivalence of (1) and (2) with the remaining conditions is more difficult; it is one of the first theorems we will prove in this course (after reviewing the definitions of all of the relevant topologies).

We are now ready to introduce the main definition of this course.

Definition 3. A *von Neumann algebra* is a $*$-subalgebra $A \subseteq B(V)$ satisfying the equivalent conditions of Theorem 2.

Remark 4. Let $A \subseteq B(V)$ be a von Neumann algebra. Then the commutant $A' \subseteq B(V)$ is also a von Neumann algebra. Moreover, A and A' determine one another (since A is also the commutant of A').

Example 5. Let $\alpha : G \rightarrow B(V)$ be a unitary representation of a group G. Then $\alpha(G)' \subseteq B(V)$ and $\alpha(G)'' \subseteq B(V)$ are von Neumann algebras.

We will later show that any von Neumann algebra $A \subseteq B(V)$ is generated by the projections which are contained in A. We may therefore reformulate $(*')$ as follows:

$(*'')$ Let $\alpha : G \rightarrow B(V)$ be a unitary representation of a group G. Describe the von Neumann algebra $\alpha(G)'$ (or, equivalently, its commutant $\alpha(G)''$).

Note that $\alpha(G)''$ is the smallest von Neumann algebra containing the subset $\alpha(G) \subseteq B(V)$. It may therefore be described as the closure (in the strong, weak, ultrastrong, or ultraweak topology) of the subalgebra of $B(V)$ generated by $\alpha(G)$ (by Theorem 2). We can therefore think of $\alpha(G)''$ as a version of the group algebra $\mathbf{C}[G]$ (after suitable completion).

Definition 6. Let G be a group equipped with a unitary representation $\alpha : G \rightarrow B(V)$. The *group von Neumann algebra* of the representation α is the double commutant $\alpha(G)''$.

Example 7. Let G be a locally compact group, so that G acts (continuously) on the Hilbert space $L^2(G)$ (by left translations). The group von Neumann algebra of the representation $\alpha : G \rightarrow B(L^2(G))$ is often called the *group von Neumann algebra of G*.

Of course, there is nothing special about $L^2(G)$ here: we can replace it by any continuous representation V of G (for example, by the direct sum of all isomorphism classes of representations of G of some bounded size).

Example 8. Let G be a finite group acting (unitarily) on a finite-dimensional Hilbert space V. In this case, we can ignore the topology on $B(V)$: the group von Neumann algebra of G is simply the image of the map $\mathbf{C}[G] \rightarrow B(V)$. In particular, it is a finite-dimensional semisimple algebra over $\mathbf{C}$: that is, a product of finitely many matrix rings $M_n(\mathbf{C})$.

Passing from the group G to the von Neumann algebra of a representation $\alpha : G \to B(V)$ generally loses a great deal of information. However, it retains the information we are interested in: namely, the structure of all G-equivariant direct sum decompositions of G. Moreover, the von Neumann algebra of α is typically a much simpler object than G itself. We have seen this already in Example 8: the classification of finite groups is very complicated, but the classification of finite-dimesional semisimple algebras over $\mathbf{C}$ is very simply. The situation in infinite dimensions is similar: there is a well-developed structure theory for von Neumann algebras, and developing that structure theory is one of our major goals in this course.

The theory of von Neumann algebras has applications in a variety of mathematical contexts. Roughly speaking, wherever there are Hilbert spaces, there will be von Neumann algebras.

Example 9. One of the original motivations for the development of the theory by Murray and von Neumann was for applications to quantum mechanics. In this setting, one has a Hilbert space V of states, and $B(V)$ (or at least the self-adjoint elements of $B(V)$) correspond to observable quantities. Two elements $f, g \in B(V)$ commute if they are simultaneously observable: that is, if measuring the quantity f does not interfere with the measurement of the quantity g. A von Neumann algebra $A \subseteq B(V)$ then corresponds to a collection of observations that can be made without disturbing some other aspect of the quantum mechanical system.

Example 10. Let X be a measure space and let $L^2(X)$ be the space of square-integrable functions on X. Then $L^2(X)$ is acted on by the space $L^\infty(X)$ of essentially bounded functions on X. This determines an embedding $L^\infty(X) \hookrightarrow B(L^2(X))$, whose image is a von Neumann algebra in $B(L^2(X))$ (the multiplication on $L^\infty(X)$ is simply given by the usual multiplication of functions).

We say that a von Neumann algebra $A \subseteq B(V)$ is *abelian* if the elements of A commute with one another, or equivalently if $A \subseteq A'$. The von Neumann algebras of Example 10 are all commutative. In fact, the converse is true as well: any abelian von Neumann A is isomorphic to $L^\infty(X)$ for some measure(able) space X. Moreover, X is canonically determined (up to a set of measure zero). We may therefore think of the theory of von Neumann algebras as a *generalization* of measure theory: it reduces to the study of measureable spaces when we restrict to the abelian case.

Example 11. Let G be a locally compact abelian group, and let $A \subseteq B(L^2(G))$ be its group von Neumann algebra. Then A is abelian. Moreover, we can identify A with $L^\infty(G^\vee)$, where $G^\vee$ is the Pontryagin dual group of G (that is, the group of continuous characters $G \to S^1$).

If $A \subseteq B(V)$ is an arbitrary von Neumann algebra, the *center* of A is the intersection $A \cap A'$. This is an abelian von Neumann algebra, hence of the form $L^\infty(X)$ for some measurable space X. We say that A is a *factor* if $A \cap A' = \mathbf{C}$. The first step in the structure theory of von Neumann algebras is to realize that a general von Neumann algebra A is naturally "distributed over" the space X. We will make this precise by constructing a family of factors $\{A_x\}_{x \in X}$ and an isomorphism of A with the "direct integral" $\int_{x \in X} A_x$. In this way, the theory of von Neumann algebras can be reduced to the study of factors.

The measurable space X appearing above has an interpretation as the space of "isotypy types" of representations of A. If A is a factor, then all representations of A can be regarded as isotypic: more precisely, if V and W are arbitrary nonzero representations of A, then V is contained in a direct sum of copies of W, and vice versa.

Theorem 12. *Let $A \subseteq B(V)$ be a factor. The following conditions are equivalent:*

(1) *The von Neumann algebra A has an irreducible representation.*

(2) *The von Neumann algebra A is isomorphic to $B(W)$, for some Hilbert space W.*

A von Neumann algebra which satisfies the equivalent conditions of Theorem 12 is said to be a factor of *type I*. (There is a further classification of the remaining factors into types II and III, which we will discuss later in this course.) If A is a type I factor, then every representation of A can be written as a direct sum of irreducible representations.

Here is an example of the kind of theorem we might aim at at the end of this course:

Theorem 13. *Let G be a reductive linear algebraic group (over the field $\mathbb{R}$ of real numbers) and let A be its group von Neumann algebra. Then A can be written as a direct integral $\int_X A_x$ of type I factors A_x. Consequently, the regular representation $L^2(G)$ can be described as a direct integral of irreducible representations.*