The Adams Spectral Sequence for MU (Lecture 9)

February 18, 2010

In this lecture, we will apply the Adams spectral sequence to obtain information about the homotopy ring $\pi_* \mathrm{MU}$. Let us begin by recalling the major conclusions of the last lecture:

(1) Let X be a commutative ring spectrum. Then $\mathrm{H}_*(X;\mathbf{F}_2)$ is a commutative $\mathbf{F}_2$-algebra; we may therefore regard $\operatorname{Spec} \mathrm{H}_*(X;\mathbf{F}_2)$ as a scheme Z over $\operatorname{Spec} \mathbf{F}_2$.

(2) The scheme $\operatorname{Spec} \mathrm{H}_*(X;\mathbf{F}_2)$ is acted on by the group scheme $\mathbb{G} = \operatorname{Spec} \mathcal{A}^\vee$ of automorphisms of the formal additive group which are equal to the identity to first order. In other words, $\mathbb{G}$ is the group scheme characterized by the formula

$$\mathrm{Hom}_{\mathbf{F}_2}(\operatorname{Spec} R, \mathbb{G}) = \{f(x) \in R[[x]] : f(x) = x + a_1 x^2 + a_2 x^4 + a_3 x^8 + \cdots\}.$$

(3) The cohomology groups $\mathrm{H}^b(Z/\mathbb{G}, \mathcal{O}_{Z/\mathbb{G}}) = \mathrm{H}^b(\mathbb{G}; \mathrm{H}_*(X;\mathbf{F}_2))$ can be identified with E_2^{*b}, the second page of the Adams spectral sequence for computing the homotopy groups $\pi_* X$.

We would like to apply this in the situation where X is the complex bordism spectrum MU. In this case, we already have a good understanding of the homology $\mathrm{H}_*(\mathrm{MU};\mathbf{F}_2)$: it can simply be identified with a polynomial ring $\mathbf{F}_2[b_1, b_2, \ldots]$ on infinitely many generators. Consequently, the scheme $Z = \operatorname{Spec} \mathrm{H}_*(\mathrm{MU};\mathbf{F}_2)$ can be described as an infinite dimensional affine space over $\mathbf{F}_2$. However, our study of formal group laws gives a more conceptual description of Z: namely, if R is any $\mathbf{F}_2$-algebra, then $\mathrm{Hom}(\operatorname{Spec} R, Z) = \mathrm{Hom}(\mathbf{F}_2[b_1, \ldots], R) = R^\infty$ can be identified with the set of formal expressions of the form $y + b_1 y^2 + b_2 y^3 + \ldots$ in the power series ring $R[[y]]$. In other words, we can identify $\mathrm{Hom}(\operatorname{Spec} R, Z)$ with the set of coordinates on the formal additive group over R, which agree with the standard coordinate y to first order.

Since $\mathbb{G}$ is the automorphism group of the formal additive group which preserves the standard coordinate to first order, there is an obvious action of $\mathbb{G}$ on the scheme Z. However, this is *not* the action described in (2) above. The natural action of $\mathbb{G}$ on the formal additive group comes from studying the action of the Steenrod algebra on the cohomology ring $\mathrm{H}^*(\mathbf{R}P^\infty;\mathbf{F}_2) \simeq \mathbf{F}_2[[x]]$. On the other hand, we can relate Z to the space of coordinates on the formal additive group by considering the cohomology ring $\mathrm{H}^*(\mathbf{C}\mathrm{P}^\infty;\mathbf{F}_2) \simeq \mathbf{F}_2[[y]]$. The power series rings $\mathbf{F}_2[[x]]$ and $\mathbf{F}_2[[y]]$ are abstractly isomorphic (even better, by an isomorphism which respects the formal group structures). However, this isomorphism isn't relevant to our picture: for example, it does not respect gradings (the coordinate x has cohomological degree 1, while the coordinate y has cohomological degree 2). Instead, they are related by the existence of a complexification $\mathbf{R}P^\infty \to \mathbf{C}\mathrm{P}^\infty$. This induces a map on cohomology rings $\mathrm{H}^*(\mathbf{C}\mathrm{P}^\infty;\mathbf{F}_2) \to \mathrm{H}^*(\mathbf{R}P^\infty, \mathbf{F}_2)$, which is given concretely by the formula $y \mapsto x^2$. Consequently, if $f(x) = x + a_1 x^2 + a_2 x^4 + \ldots$ is an R-point of $\mathbb{G}$, then f acts on $\mathrm{H}^*(\mathbf{C}\mathrm{P}^\infty;\mathbf{F}_2) \simeq \mathbf{F}_2[[y]]$ by the formula

$$f'(y) = f(x)^2 = x^2 + a_1^2 x^4 + a_2^2 x^8 + \ldots = y + a_1^2 y^2 + a_2^2 y^3 + \cdots$$

To clarify the situation, it is convenient to introduce a bit of notation. The Frobenius map $F : \mathbb{G} \to \mathbb{G}$ determines an exact sequence of group schemes

$$0 \to \ker(F) \to \mathbb{G} \xrightarrow{F} \mathbb{G}' \to 0.$$

Here $\mathbb{G}'$ denotes the group scheme $\mathbb{G}$, but regarded as the quotient $\mathbb{G}/\ker(F)$. In other words, we should think of $\mathbb{G}$ as acting on the power series ring $\mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2) \simeq \mathbf{F}_2[[x]]$, and $\mathbb{G}'$ as acting on the subring $\mathrm{H}^*(\mathbf{C}\mathrm{P}^\infty; \mathbf{F}_2) \simeq \mathbf{F}_2[[y]] \simeq \mathbf{F}_2[[x^2]]$. The action of $\mathbb{G}$ on the scheme Z factors through the Frobenius map F: in other words, it is trivial on the normal subgroup $\ker(F) \subseteq \mathbb{G}$.

To understand the Adams spectral sequence, we need to study the quotient stack $Z/\mathbb{G}$. We first consider the quotient $Z/\mathbb{G}'$.

Proposition 1. *Let $Z_0 = \operatorname{Spec} \mathbf{F}_2[b_2, b_4, b_5, b_6, b_8, \ldots]$ be the closed subscheme of Z whose R-points consist of those formal coordinates $f(y) = y + b_1 y^2 + b_2 y^3 + \cdots \in R[[x]]$ for which the coefficients b_{2^i-1} of y^{2^i} vanish for $i > 0$. Then the action of $\mathbb{G}$ on Z determines an isomorphism of schemes*

$$a : \mathbb{G}' \times Z_0 \to Z.$$

In particular, $\mathbb{G}'$ acts freely on Z, and the composition

$$Z_0 \to Z \to Z/\mathbb{G}'$$

is an isomorphism of schemes.

Proof. We must show that for every $\mathbf{F}_2$-algebra R, the map a induces a bijection on R-points. In other words, we must show that if $h(y) = y + c_1 y^2 + c_2 y^3 + c_3 y^4 + \cdots$ is an arbitrary R-point of Z, then h can be written uniquely as a composition $h(y) = (f \circ g)(y)$, where g has the form $g(y) = y + a_1 y^2 + a_2 y^4 + \cdots$ and f has the form $f(y) = y + b_2 y^3 + b_4 y^5 + b_5 y^6 + b_6 y^7 + b_8 y^9 + \cdots$. In fact, we claim that the coefficients $\{a_i, b_i\}_{i<n}$ are uniquely determined by the requirement that the equation $h(y) = (f \circ g)(y)$ holds modulo y^{n+1}. Assuming this, we note that a_n or b_n (whichever is defined) is uniquely determined by examining the y^{n+1}-coefficients of $h(y)$ and $(f \circ g)(y)$. $\square$

The quotient $Z/\mathbb{G}$ can be identified with the quotient $(Z/\ker(F))/\mathbb{G}'$. Since $\ker(F)$ acts trivially on Z, we can identify $Z/\ker(F)$ with the product $Z \times B\ker(F)$. The action of $\mathbb{G}'$ determines a composite map

$$\beta : \mathbb{G}' \times Z_0 \times B\ker(F) \hookrightarrow \mathbb{G}' \times (Z \times B\ker(F)) \simeq \mathbb{G}' \times (Z/\ker(F)) \to Z/\ker(F).$$

This is pullback of the map appearing in Proposition 1, and therefore also an isomorphism. We therefore obtain an isomorphism of stacks

$$Z/\mathbb{G} \simeq (Z/\ker(F))/\mathbb{G}' \simeq (\mathbb{G}' \times Z_0 \times B\ker(F))/\mathbb{G}' \simeq Z_0 \times B\ker(F).$$

In other words, we can identify the cohomology $\mathrm{H}^b(Z/\mathbb{G}; \mathcal{O}_{Z/\mathbb{G}})$ with the tensor product $\mathbf{F}_2[b_2, b_4, b_5, \ldots] \otimes_{\mathbf{F}_2} \mathrm{H}^b(\ker(F); \mathbf{F}_2)$. It therefore remains only to compute the cohomology of $\ker(F)$.

Unwinding the definitions, we can describe the group scheme $\ker(F)$ as follows: an R-point of $\ker(F)$ is a power series of the form

$$g(x) = x + a_1 x^2 + a_2 x^4 + \cdots$$

where $a_i^2 = 0$ for each i. It is very easy to compose such power series: if $g'(x)$ is given by $x + a'_1 x^2 + a'_2 x^4 + \cdots$, then the composition $(g' \circ g)(x)$ is given by $x + (a_1 + a'_1)x^2 + (a_2 + a'_2)x^4 + \ldots$ In other words, we can identify $\ker(F)$ with a *product* of infinitely many copies of the group scheme $\alpha_2 = \operatorname{Spec} \mathbf{F}_2[a]/(a^2)$, whose R-points are given by elements $a \in R$ such that $a^2 = 0$ (regarded as a group with respect to addition). We are therefore reduced to computing the cohomology of the group scheme α_2.

To understand this cohomology, we need to understand what it means for a vector space V to have an action of the group α_2. By definition, this is just a map $V \to V \otimes_{\mathbf{F}_2} \mathbf{F}_2[a]/(a^2)$ compatible with the comultiplication $a \mapsto a \otimes 1 + 1 \otimes a$ on $\mathbf{F}_2[a]/(a^2)$. Note that this category depends *only* on the comultiplication on $\mathbf{F}_2[a]/(a^2)$, not on its multiplication. There is an isomorphism of coalgebras

$$\theta : \mathbf{F}_2[a]/(a^2) \simeq \mathbf{F}_2^{\mathbf{Z}/2\mathbf{Z}},$$

carrying 1 to $(1,1)$ and a to $(0,1)$. It follows that the category of representations of α_2 is equivalent to the category of representations of the group $\mathbf{Z}/2\mathbf{Z}$ (note that θ is *not* an isomorphism of algebras: this means that our equivalence of categories does not respect tensor products). Under this equivalence of categories, the trivial representation V of α_2 goes to a 1-dimensional representation of $\mathbf{Z}/2\mathbf{Z}$, which must itself be trivial. It follows that we have canonical isomorphisms

$$\mathrm{H}^*(\alpha_2; \mathbf{F}_2) = \mathrm{Ext}^*(V, V) = \mathrm{H}^*(\mathbf{Z}/2\mathbf{Z}; \mathbf{F}_2) = \mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2)$$

is a polynomial algebra $\mathbf{F}_2[\epsilon]$, where ϵ has cohomological degree 1.

It follows that $\mathrm{H}^*(\ker(F); \mathbf{F}_2)$ can be identified with a polynomial ring $\mathbf{F}_2[\epsilon_1, \epsilon_2, \ldots]$, where each ϵ_i has cohomological degree 1. However, there is another grading on this cohomology ring, coming from the grading on the ring of functions $\mathbf{F}_2[a_1, a_2, \ldots]/(a_1^2, a_2^2, \ldots)$. This grading is determined by the requirement that the expression $x + a_1 x^2 + a_2 x^4 + \ldots$ has total degree -1, where x has degree -1: in other words, each a_i has degree $2^i - 1$.

We can summarize our discussion as follows:

Proposition 2. *The second page of the mod 2 Adams spectral sequence for* MU *is given by*

$$E_2^{*,*} \simeq \mathbf{F}_2[b_2, b_4, b_5, b_6, b_8, \ldots, \epsilon_1, \epsilon_2, \ldots].$$

Here each b_i has bidegree $(2i, 0)$, while each ϵ_j has bidegree $(2^j - 1, 1)$.

Note that the total degree of each of the polynomial generators in Proposition 2 is even. It follows that a group $E_2^{a,b}$ can be nonzero only when the total degree $a - b$ is even. Consequently, there can be no nontrivial differentials in the Adams spectral sequence in the second page or beyond.