The Homology of MU (Lecture 7)

February 9, 2010

Last week, we defined the complex bordism spectrum MU and showed that it was a universal complex oriented cohomology theory. In particular, there is a formal group law $f(x,y)$ over the ring $\pi_* \mathrm{MU}$. This formal group law is classified by a map $L \to \pi_* \mathrm{MU}$, where L is the Lazard ring. Our goal this week is to prove the following fundamental result:

Theorem 1 (Quillen)**.** *The map $L \to \pi_* \mathrm{MU}$ is an isomorphism. (In particular, the spectrum MU has homotopy groups only in even degrees.)*

The obstacle to overcome in the proof of Theorem 1 is that homotopy groups are typically difficult to compute. In this lecture, we will consider the much easier problem of computing the *homology* groups $\mathrm{H}_*(\mathrm{MU}; \mathbf{Z})$. In fact, we will do something a little more general: namely, we compute the homology $E_*(\mathrm{MU})$, where E is an arbitrary complex oriented cohomology theory.

Since MU is the (homotopy) colimit of the sequence $\mathrm{MU}(n)$, we have $E_*(\mathrm{MU}) \simeq \varinjlim E_*(\mathrm{MU}(n))$. Since every complex vector bundle has a canonical E-orientation, we obtain a canonical isomorphism of $E_*(\mathrm{MU}(n))$ with $E_*(\mathrm{BU}(n))$. Recall that $E_*(\mathrm{BU}(n))$ can be identified with the symmetric power $\mathrm{Sym}^n E_*(\mathrm{BU}(1)) = \mathrm{Sym}^n(\pi_* E\{\beta_0, \beta_1, \ldots\})$, where $\{\beta_i\}$ is the dual basis to topological basis $\{t^i\}$ for $E^*(BU(1)) \simeq (\pi_* E)[[t]]$. Correspondingly, we we identify $E_*(\mathrm{MU}(n))$ with the symmetric power $\mathrm{Sym}^n E_*(\mathrm{MU}(1)) \simeq \mathrm{Sym}^n(\pi_* E\{b_0, b_1, b_2, \ldots\})$, where the $\{b_i\}$ are a dual basis to the basis $\{t^{i+1}\}$ for the cohomology

$$E^*(\mathrm{MU}(1)) \simeq \widetilde{E}^*(\mathbf{C}\mathrm{P}^\infty) \simeq t(\pi_* E)[[t]] \subseteq (\pi_* E)[[t]] \simeq E^*(\mathbf{C}\mathrm{P}^\infty).$$

To pass to the bordism spectrum MU, we need to understand the transition maps $E_*(\mathrm{MU}(n)) \to E_*(\mathrm{MU}(n+1))$. These maps are induced by the composition

$$\mathrm{MU}(n) \simeq S \otimes \mathrm{MU}(n) \simeq \mathrm{MU}(0) \otimes \mathrm{MU}(n) \to \mathrm{MU}(1) \otimes \mathrm{MU}(n) \to \mathrm{MU}(n+1).$$

In the case $n = 0$, the inclusion $\mathrm{MU}(0) \to \mathrm{MU}(1)$ induces a map $\pi_* E = E_*(\mathrm{MU}(0)) \to E_*(\mathrm{MU}(1))$, which simply corresponds to the element b_0 in our chosen basis for $E_*(\mathrm{MU}(1))$. We conclude:

- For each $n \geq 0$, the map on homology $\mathrm{Sym}^n(\pi_* E)\{b_0, b_1, \ldots\} \simeq E_*(\mathrm{MU}(n)) \to E_*(\mathrm{MU}(n+1)) \simeq \mathrm{Sym}^{n+1}(\pi_* E)\{b_0, b_1, \ldots\}$ is given by multiplication by the class b_0.

There is a map of polynomial algebras $(\pi_* E)[b_0, b_1, b_2, \ldots] \to (\pi_* E)[b_1, b_2, \ldots]$ which carries b_0 to 1. This map induces an isomorphism from $\mathrm{Sym}^n(\pi_* E\{b_0, b_1, \ldots\})$ to $\mathrm{Sym}^{\leq n}(\pi_* E\{b_1, b_2, \ldots\})$. Under these isomorphisms, the map $E_*(\mathrm{MU}(n)) \to E_*(\mathrm{MU}(n+1))$ simply corresponds to the inclusion $\mathrm{Sym}^{\leq n}(\pi_* E\{b_1, b_2, \ldots\}) \hookrightarrow \mathrm{Sym}^{\leq n+1}(\pi_* E\{b_1, b_2, \ldots\})$. Passing to the limit as n grows, we obtain the following:

Proposition 2. *Let E be a complex oriented cohomology theory, and let $\{b_i\} \subseteq E_*(\mathrm{MU}(1))$ be dual to the topological basis $\{t^{i+1}\}$ for $E^*(\mathrm{MU}(1)) \simeq t(\pi_* E)[[t]]$. Then the images of the b_i in $E_*(\mathrm{MU})$ determine a ring isomorphism $(\pi_* E)[b_1, b_2, \ldots] \simeq E_*(\mathrm{MU})$ (note that the image of b_0 is the identity of $E_*(\mathrm{MU})$).*

Corollary 3. *There is a canonical isomorphism $\mathrm{H}_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, b_2, \ldots]$.*

To use this observation in the proof of Theorem 1, we need to understand the composition $L \to \pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, b_2, \ldots]$ (here the second map is the Hurewicz homomorphism). This map classifies a formal group law over the commutative ring $\mathbf{Z}[b_1, b_2, \ldots]$. We will see in a moment that this is the same formal group law that we studied in Lecture 2.

It will be convenient to again consider a slightly more general problem. Let E be any complex oriented cohomology theory. The smash product $\mathrm{MU} \otimes E$ is another multiplicative cohomology theory, with $\pi_*(\mathrm{MU} \otimes E) = E_*(\mathrm{MU}) \simeq (\pi_* E)[b_1, b_2, \ldots]$. This multiplicative cohomology theory has *two* complex orientations: one coming from our given complex orientation on E, and one from the universal complex orientation on MU. In other words, we can find two classes $t_E, t_{\mathrm{MU}} \in \widetilde{\mathrm{MU} \otimes E}^2(\mathbf{C}\mathrm{P}^\infty)$, which determine isomorphisms

$$(\pi_* E)[b_1, b_2, \ldots][[t_E]] \simeq (\mathrm{MU} \otimes E)^*(\mathbf{C}\mathrm{P}^\infty) \simeq (\pi_* E)[b_1, b_2, \ldots][[t_{\mathrm{MU}}]].$$

In particular, we can write t_{MU} as a power series

$$\sum_{i \geq 1} a_i t_E^{i+1}$$

for some coefficients $a_i \in (\pi_* E)[b_1, b_2, \ldots]$.

Claim 4. *We have $a_i = b_i$: that is, we can write $t_{\mathrm{MU}} = t_E + b_1 t_E^2 + b_2 t_E^3 + \ldots$.*

To prove the claim, note that we can think of a class in $\widetilde{\mathrm{MU} \otimes E}^2(\mathbf{C}\mathrm{P}^\infty)$ as a map of spectra $\mathrm{MU}(1) \to \mathrm{MU} \otimes E$. By general nonsense, this is the same thing as a map of E-module spectra from $\mathrm{MU}(1) \otimes E$ to $\mathrm{MU} \otimes E$. Consequently, t_E and t_{MU} correspond to a pair of maps $\phi_{\mathrm{MU}}, \phi_E : \mathrm{MU}(1) \otimes E \to \mathrm{MU} \otimes E$.

For every integer i, the class $b_i \in E_{2i}(\mathrm{MU}(1))$ determines a map of E-modules $\Sigma^{2i} E \to \mathrm{MU}(1) \otimes E$. Taking the coproduct, we obtain an equivalence of E-module spectra

$$\oplus_{i \geq 0} \Sigma^{2i} E \simeq \mathrm{MU}(1) \otimes E.$$

Consequently, to describe a map of spectra from $\mathrm{MU}(1) \otimes E$ to $\mathrm{MU} \otimes E$, we just need to specify its restriction to $\Sigma^{2i} E$ for every integer i.

The map ϕ_E is given by the composition

$$E \otimes \mathrm{MU}(1) \xrightarrow{\lambda} E \xrightarrow{u} E \otimes \mathrm{MU},$$

where λ classifies the complex orientation of E and u is the unit map $E \to E \otimes \mathrm{MU}$. Since the $\{b_i\}$ are chosen to be the dual basis to $\{t^{i+1}\}$, we see that λ vanishes on $\Sigma^{2i} E$ for $i > 0$, and restricts to the identity map $\Sigma^{2i} E \simeq E$ when $i = 0$.

The map ϕ_{MU} is given by smashing with E the canonical map $\mathrm{MU}(1) \to \mathrm{MU}$. In particular, ϕ_{MU} can be identified with the coproduct of the family of maps $\phi^i_{\mathrm{MU}} : \Sigma^{2i} E \to \mathrm{MU} \otimes E$ classified by $b_i \in E_{2i}(\mathrm{MU})$.

Note that the tensor product $\mathrm{MU}(1) \otimes E$ is acted on by the function spectrum $E^{\mathbf{C}\mathrm{P}^\infty}$: at the level of homology, this is given by the action of the cohomology ring $E^*(\mathbf{C}\mathrm{P}^\infty)$ on the reduced homology $\widetilde{E}_*(\mathbf{C}\mathrm{P}^\infty)$ (via the cap product). In particular, our complex orientation t induces a map $\Sigma^{-2}(\mathrm{MU}(1) \otimes E) \to \mathrm{MU}(1) \otimes E$, which we will denote by T. In terms of our identification $\mathrm{MU}(1) \otimes E \simeq \oplus_{i \geq 0} \Sigma^{2i} E$, the map T carries $\Sigma^{-2}\Sigma^{2i} E$ to $\Sigma^{2(i-1)} E$ by the identity map for $i > 0$, and is zero otherwise.

It follows that ϕ_{MU} can be written as a formal sum $\sum_i \phi^i_{\mathrm{MU}}$, where ϕ^i_{MU} is given by the composition

$$\mathrm{MU}(1) \otimes E \xrightarrow{T^i} \mathrm{MU}(1) \otimes E \xrightarrow{\lambda} E \xrightarrow{b_i} \mathrm{MU} \otimes E.$$

In other words, we have the formula

$$\phi_{\mathrm{MU}} = \sum_i b_i \phi_E \circ T^i.$$

Identifying ϕ_{MU} and ϕ_E with classes in $(\mathrm{MU}\otimes E)^0(\mathrm{MU}(1)) \simeq t_E(\pi_*)[b_1,\ldots][[t_E]]$, we see that T^i is given by multiplication by t_E. It follows that we have

$$t_{\mathrm{MU}} = \sum_i t_E^i(b_i t_E) = \sum_i b_i t_E^{i+1}.$$

This completes the proof of Claim 4.

Let R be the graded-commutative ring $\pi_*(\mathrm{MU}\otimes E) \simeq E_*(\mathrm{MU}) \simeq (\pi_* E)[b_1, b_2, \ldots]$. The complex orientations t_E and t_{MU} give rise to a pair of formal group laws $f_E, f_{\mathrm{MU}} \in R[x,y]$. These formal group laws can be characterized as follows: if $\pi_1, \pi_2 : \mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to \mathbf{C}\mathrm{P}^\infty$ are the two projection maps and $m : \mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to \mathbf{C}\mathrm{P}^\infty$ denotes the multiplication, then we have

$$m^* t_E = f_E(\pi_1^* t_E, \pi_2^* t_E) \qquad m^* t_{\mathrm{MU}} = f_{\mathrm{MU}}(\pi_1^* t_{\mathrm{MU}}, \pi_2^* t_{\mathrm{MU}})$$

in the cohomology ring $(\mathrm{MU}\otimes E)^*(\mathbf{C}\mathrm{P}^\infty)$. This immediately gives the following result:

Proposition 5. *Let E be a complex oriented cohomology theory and let R, $f_E, f_{\mathrm{MU}} \in R[[x,y]]$ be defined as above. Let $g(x) \in R[[x]]$ denote the power series $g(x) = x + b_1 x^2 + b_2 x^3 + \cdots$, so we have the formal identity $t_{\mathrm{MU}} = g(t_E)$. Then f_{MU} is given by the formula*

$$f_{\mathrm{MU}}(x,y) = g \circ f_E(g^{-1}(x), g^{-1}y).$$

Specializing to the case where E is an Eilenberg-MacLane spectrum $H\mathbf{Z}$, we deduce:

Corollary 6. *Let $E = \mathrm{MU}\otimes H\mathbf{Z}$, equipped with the complex orientation coming from* MU. *Then $\pi_* E \simeq H_*(\mathrm{MU}; \mathbf{Z}) \simeq \mathbf{Z}[b_1, b_2, \ldots]$, and the formal group law over $\mathbf{Z}[b_1, b_2, \ldots]$ is given by the formula $f(x,y) = g(g^{-1}(x) + g^{-1}(y))$, where $g(x) = x + b_1 x^2 + b_2 x^3 + \ldots$.*

It follows that the composition $L \to \pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z})$ is the homomorphism studied in Lecture 2. We conclude:

Corollary 7. *The composite map $L \to \pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z})$ is an isomorphism after tensoring with $\mathbf{Q}$.*

Since the Hurewicz map $\pi_* \mathrm{MU} \to \mathrm{H}_*(\mathrm{MU}; \mathbf{Z})$ is *always* a rational isomorphism, we deduce the following baby version of Theorem 1:

Corollary 8. *The map $L \to \pi_* \mathrm{MU}$ induces an isomorphism after tensoring with $\mathbf{Q}$.*

Since MU is a connective spectrum whose homology groups $\mathrm{H}_n(\mathrm{MU}; \mathbf{Z})$ are finitely generated, we conclude that the homotopy groups $\pi_n \mathrm{MU}$ are finitely generated abelian groups. Consequently, to prove Theorem 1 holds integrally, it will suffice to show that the map $L \to \pi_* \mathrm{MU}$ becomes an isomorphism after p-adic completion, for every prime number p. We will prove this later in the week, using the Adams spectral sequence.