Complex-Oriented Cohomology Theories (Lecture 4)

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In this lecture, we will introduce the notion of a *complex-oriented* cohomology theory E. We will generally not distinguish between a cohomology theory E and the spectrum that represents it. The E-cohomology groups of a space X are given by

$$E^n(X) = \pi_{-n}E^X = [X, \Omega^{\infty-n}E] = \mathrm{Hom}(\Sigma^\infty X, \Sigma^n E),$$

while the E-homology groups of X are given by $E_n(X) = \pi_n(E \otimes \Sigma^\infty X)$.

Warning 1. In this class, we will not employ the usual notations in dealing with spectra. Instead we will denote the smash product with the symbol $\otimes$, and the coproduct by $\oplus$.

We will say that a cohomology theory is *multiplicative* if its representing spectrum E is equipped with a multiplication

$$E \otimes E \to E$$

which is associative and unital up to homotopy. We will generally also assume that E is homotopy commutative, though it is sometimes convenient to relax this assumption.

Definition 2. A multiplicative cohomology theory E is *complex-orientable* if the map $E^2(\mathbf{C}\mathrm{P}^\infty) \to E^2(S^2)$ is surjective. Here we identify the 2-sphere S^2 with $\mathbf{C}\mathrm{P}^1 \subseteq \mathbf{C}\mathrm{P}^\infty$.

We will henceforth regard S^2 and $\mathbf{C}\mathrm{P}^\infty$ as pointed spaces. A choice of base point gives canonical decompositions

$$E^2(\mathbf{C}\mathrm{P}^\infty) \simeq \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty \oplus E^2(*) \qquad E^2(S^2) \simeq \widetilde{E}^2(S^2) \oplus E^2(*);$$

here the $\widetilde{E}$ denotes reduced cohomology with coefficients in E. Note that $\widetilde{E}^2(S^2) \simeq E^0(*) \simeq \pi_0 E$ is equipped with a canonical unit element $\bar{t}$. Since the image of the map $\theta : \widetilde{E}(\mathbf{C}\mathrm{P}^\infty) \to \widetilde{E}^2(S^2)$ is a $(\pi_0 E)$-module, θ is surjective if and only if its image contains $\bar{t}$. In other words:

- A multiplicative cohomology theory E is complex-orientable if and only if there exists an element $t \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty)$ such that $\theta(t) = \bar{t}$ is the canonical generator of $\widetilde{E}^2(S^2)$.

We will refer to a choice of $t \in \widetilde{E}^2(\mathbf{C}\mathrm{P}^\infty) \subseteq E^2(\mathbf{C}\mathrm{P}^\infty)$ as a *complex orientation* of E.

Remark 3. Let E be a multiplicative cohomology theory and let E' be its connective cover. Then the canonical map $\widetilde{E'}^2(X) \to \widetilde{E}^2(X)$ is an isomorphism whenever X is simply connected. It follows that E is complex orientable if and only if E' is complex-orientable: better yet, there is a bijection between complex orientations of E and complex orientations of E'.

Remark 4. We can think of $\bar{t}$ as encoding a pointed map $S^2 \to \Omega^\infty E$. A complex orientation of E is an extension of this map to $\mathbf{C}\mathrm{P}^\infty$. The existence of such a map can often be established by obstruction theory. For example, if we are already given an extension of $\bar{t}$ to $\mathbf{C}\mathrm{P}^n$, then there is an obstruction to further extending to $\mathbf{C}\mathrm{P}^{n+1}$ which lies in the homotopy group $\pi_{2n+1}\Omega^\infty E = \pi_{2n+1}E = E^{-2n-1}(*)$. In particular, if we have $\pi_3 E = \pi_5 E = \ldots$, then E is complex-orientable.

Example 5. Ordinary cohomology (with coefficients in any commutative ring R) is complex-orientable. In fact, the restriction map $\mathrm{H}^2(\mathbf{CP}^\infty; R) \to \mathrm{H}^2(S^2; R)$ is an isomorphism.

Example 6. Complex K-theory is complex-orientable. This follows from Remark 4, since $\pi_i K = 0$ whenever i is odd. In this case, the complex orientation is not unique. However, there is a canonical complex orientation, given by the class $t \in K^2(\mathbf{CP}^\infty) \simeq K^0(\mathbf{CP}^\infty) = [\mathcal{O}(1)] - 1$, where the first map is Bott periodicity and $\mathcal{O}(1)$ denotes the universal complex line bundle on $\mathbf{CP}^\infty$.

We next show that the existence of a complex orientation on E often forces the Atiyah-Hirzebruch spectral sequence for E to degenerate. We begin with a degeneration criterion (not the most general, but sufficient for our purposes).

Proposition 7. *Let X be a space and assume that each of the homology groups $\mathrm{H}_n(X; \mathbf{Z})$ is a free abelian group on generators $\{h_{\alpha,n}\}_{\alpha \in B_n}$. Let $c_{\alpha,n} \in \mathrm{H}^n(X; \mathbf{Z}) \simeq \mathrm{Hom}(\mathrm{H}_n(X; \mathbf{Z}), Z)$ be defined by the formula*

$$c_{\alpha,n}(h_{\beta,n}) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

Let E be a multiplicative cohomology theory and let $\tau_{\leq 0} E$ denote its truncation, so that $\pi_i \tau_{\leq 0} E = \begin{cases} \pi_i E & \text{if } i \leq 0 \\ 0 & \text{otherwise.} \end{cases}$. The unit $S \to E$ determines a map of spectra $H\mathbf{Z} \simeq \tau_{\leq 0} S \to \tau_{\leq 0} E$. Under this map, the homology classes $h_{n,\alpha}$ have images $h'_{n,\alpha} \in (\tau_{\leq 0} E)_n(X)$ and the cohomology classes $c_{n,\alpha}$ have images $c'_{n,\alpha} \in (\tau_{\leq 0} E)^n(X)$. Assume that one of the following conditions is satisfied:

- $(*)$ *Each of the homology classes $h'_{n,\alpha}$ can be lifted to a class $h''_{n,\alpha} \in E_n(X)$.*
- $(*')$ *Each of the groups $\mathrm{H}_n(X; \mathbf{Z})$ is finitely generated, and each of the cohomology classes $c'_{n,\alpha}$ can be lifted to a class $c''_{n,\alpha} \in E^n(X)$.*

Then:

1. *The smash product $E \otimes \Sigma^\infty X_+$ is equivalent, as an E-module, to a coproduct $\bigoplus_{n, \alpha \in B_n} \Sigma^n E$.*
2. *The function spectrum E^X is equivalent to a product $\prod_{n, \alpha \in B_n} \Sigma^{-n} E$.*
3. *We have (noncanonical) isomorphisms $E_*(X) \simeq \pi_* E \otimes \mathrm{H}_*(X)$ and $E^*(X) \simeq \mathrm{Hom}(\mathrm{H}_*(X), \pi_*(E))$.*

Proof. We will prove (1); assertions (2) and (3) are obvious consequences. Let Y denote the suspension spectrum $\Sigma^\infty X_+$. In what follows, we will not use that Y is a suspension spectrum: only that Y is connective with freely generated homology. We construct a sequence of spectra

$$Y_0 \to Y_1 \to \dots$$

having colimit Y, with the following additional properties:

- (a) The map $Y_n \to Y$ induces an isomorphism in homology in degrees $\leq n$. In particular, Y is homotopy equivalent to the colimit of the sequence $\{Y_n\}$.
- (b) The spectrum Y_n is build from finitely many spheres of dimension $\leq n$; in particular, the cohomology groups $\mathrm{H}^k(Y_n; \mathbf{Z})$ vanish for $k > n$.

Assume that Y_{n-1} has been constructed, and let Z_n denote the cofiber of the map $Y_{n-1} \to Y$. Then Z_n is $(n-1)$-connected, and the map $\mathrm{H}_n(Y; \mathbf{Z}) \to \mathrm{H}_n(Z_n; \mathbf{Z})$ is an isomorphism. By the Hurewicz theorem, the image of each of the homology classes $h_{n,\alpha}$ is represented by a map $S^n \to Z_n$. Let $Z'_n = \bigoplus_{\alpha \in B_n} S^n$ and let $\phi_n : Z'_n \to Z_n$ be the induced map, so that we have a cofiber sequence

$$Z'_n \to Z_n \to Z''_n.$$

We now define Y_n to be the homotopy fiber product $Y \times_{Z_n} Z'_n$; in other words, Y_n is the homotopy fiber of the composite map $Y \to Z_n \to Z''_n$. It is easy to see that (a) and (b) hold.

Now suppose that $(*)$ is satisfied. Each $h''_{n,\alpha}$ is represented by a map of E-modules $\Sigma^n E \to E \otimes Y$. We will prove:

(c) The map $\theta : \bigoplus_{n,\alpha \in B_n} \Sigma^n E \to E \otimes Y$ is a homotopy equivalence.

To prove (c), it suffices to show that θ is k-connected for every value of k. This is obvious for $k = 0$. Assume that $k > 0$. Note that that ϕ_0 induces an E-module map $\bigoplus_{\alpha \in B_0} E \simeq E \otimes Z'_0 \to E \otimes Y$, which we can identify with a sequence of homology classes $b_{\alpha,0} \in E_0(Y)$. By construction, the classes $b_{\alpha,0}$ lift the classes $h'_{\alpha,0}$. Since Y is connective, we have $(\tau_{\geq 1} E)_0(Y) \simeq 0$ so that the map $E_0(Y) \to (\tau_{\leq 0} E)_0(Y)$ is injective; it follows that $b_{\alpha,0} = h''_{\alpha,0}$. We therefore have a map of cofiber sequences

$$\begin{tikzcd}
\bigoplus_{\alpha \in B_0} E \arrow[r] \arrow[d, "\theta'"] & \bigoplus_{n,\alpha \in B_n} \Sigma^n E \arrow[r] \arrow[d, "\theta"] & \bigoplus_{n>0,\alpha \in B_n} \Sigma^n E \arrow[d, "\theta''"] \\
Z'_0 \arrow[r] & Y \arrow[r] & Z''_0.
\end{tikzcd}$$

Since θ is a homotopy equivalence, to prove that θ' is k-connective it suffices to show that θ'' is k-connective. This follows from the inductive hypothesis, applied to the connective spectrum $\Sigma^{-1} Z''_0$.

Now suppose that condition $(*')$ is satisfied. We will prove, using induction on n, that each of the maps $E \otimes Y \to E \otimes Z_n$ admits a splitting $s_n : E \otimes Z_n \to E \otimes Y$, so that the cohomology classes $c''_{\alpha,n}$ give maps

$$\phi_\alpha : Z_n \to E \otimes Z_n \to E \otimes Y \stackrel{c''_{\alpha,n}}{\to} \Sigma^n E.$$

Using (b), we deduce that the map $(\tau_{\leq 0} E)^n Z_n \to (\tau_{\leq 0} E)^n Y$ is injective, so each the image of $\psi_\alpha \in E^n(Z_n) \to (\tau_{\leq 0} E)^n(Z_n)$ coincides with the image of $c_{\alpha,n} \in \mathrm{H}^n(Y; \mathbf{Z}) \simeq \mathrm{H}^n(Z_n; Z) \to (\tau_{\leq 0} E)^n(Z_n)$.

Assume that s_{n-1} has been constructed. The maps $\{\psi_\alpha\}_{\alpha \in B_{n-1}}$ together yield a map $Z_n \to \bigoplus_\alpha \Sigma^n E \simeq E \otimes Z'_n$, which we can identify with an E-module map $s_n : E \otimes Z_n \to E \otimes Z'_n$. Moreover, the compatibility of the classes ϕ_α with c_{α_n} shows that the composition

$$E \otimes Z'_n \stackrel{\psi}{\to} E \otimes Z_n \stackrel{\phi}{\to} E \otimes Z'_n$$

is the identity; that is, s_n is a splitting of the projection $E \otimes Y \to E \otimes Z_n$.

It now follows that $E \otimes Y \simeq \varinjlim (E \otimes Y_n) \simeq \varinjlim_n \bigoplus_{m \leq n} E \otimes Z'_m$. □

Example 8. Let $X = \mathbf{C}\mathrm{P}^n$, and let $t \in E^2(X)$ be a complex orientation on a multiplicative cohomology theory E. Then the cohomology classes $\{1, t, t^2, \ldots, t^n\}$ satisfy the hypotheses of Proposition 7. It follows that the classes $1, t, t^2, \ldots, t^n$ form a basis for $E^*(\mathbf{C}\mathrm{P}^n)$ over $\pi_* E$. We claim that $t^{n+1} = 0$. To prove this, we may replace E by its connective cover and thereby assume that E is connective: then $t^{n+1} \in E^{2n+2}(\mathbf{C}\mathrm{P}^n)$ vanishes since $\mathbf{C}\mathrm{P}^n$ has dimension $< 2n+2$. It follows that we have a ring isomorphism $E^*(\mathbf{C}\mathrm{P}^n) \simeq (\pi_* E)[t]/(t^{n+1})$. Writing $\mathbf{C}\mathrm{P}^\infty = \varinjlim \mathbf{C}\mathrm{P}^n$, we get

$$E^*(\mathbf{C}\mathrm{P}^\infty) = \varprojlim E^*(\mathbf{C}\mathrm{P}^n) \simeq \varprojlim (\pi_* E)[t]/(t^{n+1}) \simeq (\pi_* E)[[t]].$$

Here the potential $\lim^1$-terms vanish because the maps $(\pi_* E)[t]/(t^{n+1}) \to (\pi_* E)[t]/(t^{m+1})$ are surjective.

Example 9. If $X = \mathbf{C}\mathrm{P}^m \times \mathbf{C}\mathrm{P}^n$, the same reasoning gives an isomorphism $E^*(X) \simeq (\pi_* E)[x,y]/(x^{m+1}, y^{n+1})$. Passing to the limit as before, we get an isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty) = (\pi_* E)[[x,y]]$.

The space $\mathbf{C}\mathrm{P}^\infty$ is an Eilenberg-MacLane space $K(\mathbf{Z}, 2)$, and can therefore be realized as a topological abelian group. In fact, it is easy to realize $\mathbf{C}\mathrm{P}^\infty$ as a topological monoid: we can define $\mathbf{C}\mathrm{P}^\infty$ to be the projectivization $(V - \{0\})/\mathbf{C}^*$ for any complex vector space V of infinite dimension. Taking V to be the underlying vector space of the ring $\mathbf{C}[x]$, we get a commutative and associative multiplication on $\mathbf{C}\mathrm{P}^\infty$. The multiplication map

$$\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty \to \mathbf{C}\mathrm{P}^\infty$$

classifies the operation of forming tensor products of complex line bundles. If E is a complex-oriented cohomology theory, then we get a pullback map on E-cohomology

$$(\pi_* E)[[t]] \simeq E^*(\mathbf{C}\mathrm{P}^\infty) \to E^*(\mathbf{C}\mathrm{P}^\infty \times \mathbf{C}\mathrm{P}^\infty) \simeq (\pi_* E)[[x, y]].$$

We let $f(x, y) \in (\pi_* E)[[x, y]]$ denote the image of t under this map. (The map is entirely determined by $f(x, y)$, since it is continuous with respect to the "inverse limit" topologies on the power series rings in question.)

The associativity and commutativity of the multiplication $\mathbf{C}\mathrm{P}^\infty$ imply the following:

Proposition 10. *Let E be a complex-oriented multiplicative cohomology theory. Then the above construction determines a formal group law $f(x, y) \in R[[x, y]]$, where R is the commutative ring $\oplus_n \pi_{2n} E$. This formal group law is compatible with the natural grading of R: that is, the expression $f(x, y)$ has degree -2, if we let x and y have degree -2.*

We close by describing another application of Proposition 7. Fix an integer $n \geq 0$, and let $X = BU(n)$ be the classifying space of the unitary group $U(n)$. There is a canonical map

$$\theta : (\mathbf{C}\mathrm{P}^\infty)^n \simeq BU(1) \times \cdots \times BU(1) \to BU(n).$$

This map classifies the construction $(\mathcal{L}_1, \ldots, \mathcal{L}_n) \mapsto \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, which takes the direct sum of a collection of complex line bundles. Since the formation of direct sums is commutative up to isomorphism, the map θ is Σ_n-equivariant, up to homotopy. It therefore induces a map $\mathrm{H}^*(BU(n); \mathbf{Z}) \to \mathrm{H}^*((\mathbf{C}\mathrm{P}^\infty)^n; \mathbf{Z}) \simeq \mathbf{Z}[t_1, \ldots, t_n]$, whose image is contained in the subgroup $\mathbf{Z}[t_1, \ldots, t_n]^{\Sigma_n}$ of symmetric polynomials in n-variables. This ring of invariants is given by $\mathbf{Z}[c_1, c_2, \ldots, c_n]$, where c_i is the ith elementary symmetric function on $(t_1, \ldots, t_n)$. In fact, this construction yields an isomorphism $\mathrm{H}^*(BU(n), \mathbf{Z}) \to \mathbf{Z}[c_1, \ldots, c_n]$; under this isomorphism, the cohomology class c_i corresponds to the ith Chern class of the universal bundle.

Dually, can write $\mathrm{H}_*(\mathbf{C}\mathrm{P}^\infty; \mathbf{Z}) = \mathbf{Z}\{\beta_0, \beta_1, \ldots\}$, where $\{\beta_i\}$ is the dual basis to $\{t_i\}$. Then $\mathrm{H}_*(\mathrm{BU}(n); \mathbf{Z})$ is given by $\mathrm{H}_*(\mathbf{C}\mathrm{P}^\infty; \mathbf{Z})^{\otimes n}_{\Sigma_n} = \mathrm{Sym}^n \, \mathrm{H}_*(\mathbf{C}\mathrm{P}^\infty; \mathbf{Z})$. In particular, it is a free $\mathbf{Z}$-module whose generators can be lifted to $\mathrm{H}_*((\mathbf{C}\mathrm{P}^\infty)^n; \mathbf{Z})$.

Let E be a complex-oriented multiplicative cohomology theory. Then we have a canonical isomorphism $E^*(\mathbf{C}\mathrm{P}^\infty) \simeq (\pi_* E)[[t]]$. The (topological) basis $\{t^i\}$ for this cohomology has a dual basis $\{\beta_i\}$ for $E_*(\mathbf{C}\mathrm{P}^\infty)$ over $\pi_* E$. Using the map θ, we get homology classes $\{\beta_{i_1} \beta_{i_2} \ldots \beta_{i_n}\}_{i_1 \leq \cdots \leq i_n}$ in $E_*(BU(n))$ which lift the corresponding basis for the $\mathbf{Z}$-homology of $BU(n)$. It follows from Proposition 7 that $E_*(BU(n))$ is freely generated by the classes $\{\beta_{i_1} \beta_{i_2} \ldots \beta_{i_n}\}_{0 \leq i_1 \leq \cdots \leq i_n}$ over $\pi_* E$.

The same argument shows that $E_*(BU(n) \times BU(n))$ is given by $E_*(BU(n)) \otimes_{\pi_* E} E_*(BU(n))$. The diagonal map $BU(n) \to BU(n) \times BU(n)$ determines a comultiplication on $E_*(BU(n))$. When $n = 0$, this comultiplication is dictated by the structure of the multiplication on $E^*(BU(1)) = (\pi_* E)[[t]]$: namely, it is given by $\delta_{\beta_n} = \sum_{i+j=n} \beta_i \otimes \beta_j$. Since θ induces a map of coalgebras $E_*(BU(1)^n) \to E_*(BU(n))$, this completely determines the comultiplication on $E_*(BU(n))$. More informally, we can say that the comultiplication on $E_*(BU(n))$ is given by the same formulas as in the case of integral homology. It follows that multiplication on $E^*(BU(n))$ can be described in the same way as the multiplication on the ordinary cohomology of $E^*(BU(n))$. More precisely, we have a canonical isomorphism

$$E^*(BU(n)) \simeq (\pi_* E)[[c_1, \ldots, c_n]]$$

where c_i is dual to β_1^i (with respect to the basis consisting of monomials in the β_i). We can think of the c_i as analogues of the Chern classes in E-cohomology.