Monochromatic Layers (Lecture 34)

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Fix a prime number p. To any spectrum X, we can associate its chromatic tower

$$\cdots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X.$$

If X is a finite p-local spectrum, then the chromatic convergence theorem tells us that the homotopy limit of this tower is X. In particular, we can associate to X the *chromatic spectral sequence* $\{E_r^{p,q}, d_r\}$, where $E_1^{p,*}$ is given by the homotopy groups of the homotopy fiber of the map $L_{E(p)}X \to L_{E(p-1)}X$. (In fact, the proof of the chromatic convergence theorem tells us that this spectral sequence converges in a strong sense: for example, the chromatic filtration on each homotopy group $\pi_n X$ is finite). This motivates the following:

Definition 1. For each spectrum X, we let $M_n(X)$ denote the homotopy fiber of the map $L_{E(n)}X \to L_{E(n-1)}X$. We will refer to $M_n(X)$ as the *nth monochromatic layer of* X.

The essential features of $M_n(X)$ are captured by the following definition:

Definition 2. A spectrum X is *monochromatic of height* n if it is $E(n)$-local and $E(n-1)$-acyclic.

Remark 3. For any spectrum X, we have a map of $E(n)$-local spectra $L_{E(n)}X \to L_{E(n-1)}X$ which induces an isomorphism on $E(n-1)$-homology. It follows that the fiber $M_n(X)$ is monochromatic of height n. Conversely, if X is monochromatic of height n, then $L_{E(n)}X \simeq X$ and $L_{E(n-1)}X \simeq 0$, so that $X \simeq M_n(X)$.

Example 4. Let X be a finite p-local spectrum of type $\geq n$. Then $L_{E(n)}X$ is monochromatic of height n. To see this, it suffices to observe that $E(n-1)_* L_{E(n)}X \simeq E(n-1)_* X \simeq 0$.

Notation 5. Let $\mathcal{M}_n$ denote the collection of all spectra which are monochromatic of height n. Since $L_{E(n)}$ is a smashing localization, we see that $\mathcal{M}_n$ is closed under homotopy colimits. We say that that an object $X \in \mathcal{M}_n$ is *compact* if, for every filtered diagram $\{Y_\alpha\}$ of objects of $\mathcal{M}_n$, the induced map

$$\varinjlim \mathrm{Map}(X, Y_\alpha) \to \mathrm{Map}(X, \varinjlim Y_\alpha)$$

is a homotopy equivalence.

Example 6. Let X be a finite p-local spectrum of type $\geq n$. Then $L_{E(n)}X$ is a compact object of $\mathcal{M}_n$. To see this, we note that if $\{Y_\alpha\}$ is a filtered diagram in $\mathcal{M}_n$, then we have

$$\mathrm{Map}(L_{E(n)}X, \varinjlim Y_\alpha) \simeq \mathrm{Map}(X, \varinjlim Y_\alpha) \simeq \varinjlim \mathrm{Map}(X, Y_\alpha) \simeq \varinjlim \mathrm{Map}(L_{E(n)}X, Y_\alpha).$$

Our next goal is to establish a converse to Example 6. The essential observation is the following:

Proposition 7. *Let X be a spectrum which is monochromatic of height n. Then X can be written as a filtered colimit $\varinjlim X_\alpha$, where each X_α is the $E(n)$-localization of a finite spectrum of type $\geq n$.*

Proof. We have a cofiber sequence

$$X' \to X \to L_{n-1}^t X,$$

where X' is a filtered colimit of p-local finite spectra of type $\geq n$. This induces a cofiber sequence

$$L_{E(n)} X' \to L_{E(n)} X \to L_{E(n)} L_{n-1}^t X.$$

Since $X \in \mathcal{M}_n$ we have $L_{E(n)} X \simeq 0$, and since $L_{E(n)}$ is smashing we conclude that $L_{E(n)} X'$ is a filtered colimit of $E(n)$-localizations of finite p-local spectra of type $\geq n$. It will therefore suffice to show that $L_{E(n)} L_{n-1}^t X \simeq 0$; that is, that $L_{n-1}^t X$ is $E(n)$-acyclic. Since $E(n)$ is Landweber exact, it will suffice to show that $L_{n-1}^t X$ is MU-acyclic. In the last lecture, we saw that

$$\mathrm{MU}_* L_{n-1}^t X \simeq \mathrm{MU}_* L_{E(n-1)} X,$$

and the right hand side vanishes since X is assumed to be $E(n-1)$-acyclic. □

Corollary 8. *An object $X \in \mathcal{M}_n$ is compact if and only if it is a retract of $L_{E(n)} Y$ for some finite spectrum Y of type $\geq n$.*

Proof. Write X as a filtered colimit of spectra X_α of the form $L_{E(n)} Y_\alpha$. Since X is compact, the identity map $X \to \varinjlim X_\alpha$ factors through some X_α, so that X is a retract of $L_{E(n)} Y_\alpha$. □

Corollary 9. *The homotopy theory $\mathcal{M}_n$ is compactly generated: that is, every object of $\mathcal{M}_n$ can be obtained as a filtered colimit of compact objects of $\mathcal{M}_n$.*

We want to draw attention to a crucial features of the compact objects of $\mathcal{M}_n$. First, we state a slightly stronger version of the periodicity theorem of Lecture 27:

Theorem 10. *Let X be a finite p-local spectrum of type $\geq n$. Then there exists a v_n-self map $f : \Sigma^k X \to X$ where $k = 2(p^n - 1)p^N$ for $N \gg 0$, which acts by multiplication by $v_n^{p^N}$ on $K(n)_* X$.*

Corollary 11. *Let X be a compact object of $\mathcal{M}_n$. Then X is periodic. More precisely, for $N \gg 0$, there is a homotopy equivalence $X \simeq \Sigma^{2p^N(p^n-1)} X$.*

Proof. According to Corollary 8, we can assume that X is a retract of $L_{E(n)} Y$ for some finite p-local spectrum Y of type $\geq n$. Let $f : \Sigma^k Y \to Y$ be the v_n-self map of Theorem 10, where $k = 2p^N(p^n - 1)$. Then the action of f on $K(n)_* L_{E(n)} Y \simeq K(n)_* Y$ is given by $v_n^{p^N}$. It follows that the composite map

$$f' : \Sigma^k X \to \Sigma^k L_{E(n)} Y \xrightarrow{f} L_{E(n)} Y \to X$$

induces multiplication by $v_n^{p^N}$ on $K(n)_* X$; in particular, it is bijective. Since f' is also bijective on $K(m)_* X$ for $m < n$ (since these groups vanish), we conclude that the homotopy fiber of f' is $K(m)$-acyclic for $m \leq n$ and therefore $E(n)$-acyclic. Since X is $E(n)$-local, the homotopy fiber of f' is also $E(n)$-local and therefore trivial; this proves that f' is an equivalence $\Sigma^k X \simeq X$. □

If X is a general monochromatic spectrum of height n, then X is a filtered colimit of compact objects X_α, each of which is periodic of some period $2(p^n - 1)p^{N_\alpha}$. The exponent N_α generally depends on α, so that X itself is not periodic. Nevertheless, elements of the homotopy of X are organized into "periodic families": that is, any class $x \in \pi_k X$ is given by an element in some $\pi_k X_\alpha$, which in turn determines elements of $\pi_{k+2m(p^n-1)p^{N_\alpha}} X$ for all $m \in \mathbf{Z}$. This is the motivation for the term "chromatic homotopy theory": the chromatic tower of a spectrum X is like a prism, which separates X into "monochromatic layers" $M_n(X)$ each of which exhibit a sort of generalized $2(p^n - 1)$-fold periodicity.

We conclude with a few remarks relating the monochromatic category $\mathcal{M}_n$ with the $K(n)$-local homotopy category.

Proposition 12. *The constructions*

$$X \mapsto L_{K(n)}X$$
$$Y \mapsto M_n(Y)$$

determine mutually inverse equivalences between the homotopy category of monochromatic spectra of height n and the homotopy category of $K(n)$-local spectra.

We first recall a fact we proved earlier:

Lemma 13. *Let X be an $E(n-1)$-local spectrum. Then $K(n)_*X \simeq 0$.*

Proof. Since $L_{E(n-1)}$ is smashing, $K(n) \otimes X$ is $E(n-1)$-local. It will therefore suffice to show that $K(n) \otimes X$ is $E(n-1)$-acyclic; that is, that $E(n-1) \otimes K(n) \otimes X \simeq 0$. This is clear, since $E(n-1) \otimes K(n)$ is a complex orientable spectrum whose formal group has height $< n$ and exactly n, and therefore $E(n-1) \otimes K(n) \simeq 0$. □

Proof of Proposition 12. We argue that both composite functors are the identity. First, fix a monochromatic spectrum X of height n. We wish to show that $X \simeq M_n(L_{K(n)}X)$. Since $L_{K(n)}X$ is $K(n)$-local, it is $E(n)$-local; thus $M_n(L_{K(n)}X)$ can be identified with the homotopy fiber F of the map $L_{K(n)}X \to L_{E(n-1)}L_{K(n)}X$. Since X is monochromatic, $L_{E(n-1)}X \simeq 0$ so there is a canonical map $\alpha : X \to F$. We claim that α is an equivalence. Since X and F are both $E(n)$-local, it will suffice to show that α induces an isomorphism $K(m)_*X \to K(m)_*F$ for $m \leq n$. If $m < n$, then both groups vanish. If $m = n$, we are reduced to proving that

$$K(n) \otimes X \to K(n) \otimes L_{K(n)}X \to L_{E(n-1)}L_{K(n)}X$$

is a fiber sequence. This follows from the observation that the first map is an equivalence and the third term vanishes (Lemma **??**).

Now let Y be a $K(n)$-local spectrum. Then Y is $E(n)$-local, so that $M_n(Y)$ is the homotopy fiber of the map $Y \to L_{E(n-1)}Y$. We wish to prove that the map $M_n(Y) \to Y$ exhibits Y as a $K(n)$-localization of $M_n(Y)$. Since Y is $K(n)$-local, it suffices to show that this map is a $K(n)$-equivalence; that is, that $K(n)_*L_{E(n-1)}Y \simeq 0$; this also follows from Lemma **??**. □

Corollary 14. *The $K(n)$-local stable homotopy category is compactly generated; its compact objects are precisely the retracts of spectra of the form $L_{K(n)}X$, where X is a finite spectrum of type $\geq n$.*

Warning 15. For a general finite spectrum X, the localization $L_{K(n)}X$ is not a compact object of the $K(n)$-local stable homotopy category. For example, if $n > 0$, then the $K(n)$-local sphere $L_{K(n)}S$ is not a compact object of the $K(n)$-local stable homotopy category.