Complex Bordism and $E(n)$-Localization (Lecture 33)

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Our first goal in this lecture is to complete the proof of chromatic convergence theorem by verifying the following:

Proposition 1. *For each n, let $C_n(S_{(p)})$ denote the homotopy fiber of the localization map $S_{(p)} \to L_{E(n)} S_{(p)}$. Then the maps $\mathrm{MU}_* C_n(S_{(p)}) \to \mathrm{MU}_* C_{n-1}(S_{(p)})$ are equal to zero.*

We will prove this result by explicitly computing the complex bordism of each $C_n S_{(p)}$. First, let us establish a bit of notation. Let $L = \pi_* \mathrm{MU}$ be the Lazard ring. For each $n \geq 0$, let $I(n)$ denote the ideal generated by $(v_0, v_1, \ldots, v_{n-1}, v_n)$ in L. We will say that an L-module M is *$I(n)$-torsion* if every every element $x \in M$ is annihilated by some power of the ideal $I(n)$. The basis for our calculation is the following.

Proposition 2. *Let X be an MU-module spectrum whose homotopy groups $\pi_* X$ are an $I(n-1)$-torsion module, and let $n > 0$. Let $X[v_n^{-1}]$ be the spectrum obtained by inverting $v_n \in \pi_{2(p^n-1)} \mathrm{MU}$. Then the map $X \to X[v_n^{-1}]$ exhibits $X[v_n^{-1}]$ as an $E(n)$-localization of X.*

Lemma 3. *Let Y be an MU-module spectrum such that $\pi_* Y$ is an $I(n)$-torsion L-module. Then Y is $E(n)$-acyclic.*

Proof. We have $\mathrm{MU}_* Y \simeq (\mathrm{MU}_* \mathrm{MU}) \otimes_{\pi_* \mathrm{MU}} \pi_* Y$. Note that the two maps $\phi_1, \phi_2 : L \to \mathrm{MU}_* \mathrm{MU}$ carry $I(n)$ to the same ideal, since the condition that a formal group be of height $> n$ does not depend on the choice of coordinate. It follows that $\mathrm{MU}_* Y$ is an $I(n)$-torsion L-module. Since $E(n)$ is Landweber exact, we get $E(n)_* Y \simeq \pi_* E(n) \otimes_L \mathrm{MU}_* Y$. Since $I(n)$ generates the unit ideal in $E(n)$ (the formal group associated to $E(n)$ has height $\leq n$), we conclude that every element of $E(n)_* Y$ is generated by a power of the unit ideal in $\pi_* E(n)$: that is, $E(n)_* Y \simeq 0$. □

Proof of Proposition . We must show two things:

(1) The spectrum $X[v_n^{-1}]$ is $E(n)$-local.

(2) The map $X \to X[v_n^{-1}]$ is an equivalence in $E(n)$-homology.

To prove (1), we observe that $X[v_n^{-1}]$ is a module spectrum for $\mathrm{MU}_{(p)}[v_n^{-1}]$, and therefore $E(n)$-local since $E(n)$ is Bousfield equivalent to $\mathrm{MU}_{(p)}[v_n^{-1}]$.

To prove (2), it suffices to show that the homotopy fiber of the map $X \to X[v_n^{-1}]$ is $E(n)$-acyclic. This homotopy fiber is a filtered colimit of the cofibers of maps

$$X \xrightarrow{v_n^k} \Sigma^{-2k(p^n-1)} X.$$

It therefore suffices to show that the homotopy fibers of each of these maps is $E(n)$-acyclic. Denote such a homotopy fiber by Y; then Y is an MU-module such that $\pi_* Y$ is $I(n)$-torsion, so that Y is $E(n)$-acyclic by Lemma 3. □

Proposition 4. *For each $n \geq 0$, let M_n be the L-module given by the quotient of $L_{(p)}[v_0^{-1}, \ldots, v_n^{-1}]$ by the submodules $\{L_{(p)}[v_0^{-1}, \ldots, v_{i-1}^{-1}, v_{i+1}^{-1}, \ldots, v_n^{-1}]\}_{0 \leq i \leq n}$. Then there are canonical isomorphisms $M(n) \simeq \mathrm{MU}_* C_n S_{(p)}$.*

Remark 5. The L-modules $M(n)$ can be described by recursion: we have $M(-1) \simeq L_{(p)}$, and for $n \geq 0$ there is an isomorphism $M(n) \simeq M(n-1)[v_n^{-1}]/M(n-1)$.

Proof. We use induction on n, beginning with the case $n = 0$. Note that $C_0 S_{(p)}$ is the fiber of the map $S_{(p)} \to L_{E(0)} S_{(p)} = S_{\mathbf{Q}}$. It follows that $\mathrm{MU} \otimes C_0 S_{(p)}$ is the fiber of the map $\mathrm{MU}_{(p)} \to \mathrm{MU}_{\mathbf{Q}}$. This map is injective on homotopy, so we get a short exact sequence

$$0 \to \pi_* \mathrm{MU}_{(p)} \to \pi_* \mathrm{MU}_{\mathbf{Q}} \to \mathrm{MU}_* C_0 S_{(p)} \to 0,$$

giving the isomorphism $\mathrm{MU}_* C_0 S_{(p)} \simeq L_{(p)}[p^{-1}]/L_{(p)}$.

The general case is similar. We have a fiber sequence

$$C_{n-1} S_{(p)} \to S_{(p)} \to L_{E(n-1)} S_{(p)}.$$

Note that $L_{E(n-1)} S_{(p)}$ is already $E(n)$-local, so that $C_n(L_{E(n-1)} S_{(p)}) \simeq 0$. Applying C_n, we deduce that the map $C_n C_{n-1} S_{(p)} \to C_n S_{(p)}$ is an equivalence. In other words, we have a fiber sequence

$$C_n S_{(p)} \to C_{n-1} S_{(p)} \to L_{E(n)} C_{n-1} S_{(p)}.$$

The inductive hypothesis implies that $\mathrm{MU}_*(C_{n-1} S_{(p)})$ is an $I(n-1)$-torsion L-module. It follows from Proposition that

$$\mathrm{MU}_* L_{E(n)} C_{n-1} S_{(p)} \simeq MU_*(C_{n-1} S_{(p)})[v_n^{-1}] \simeq M(n-1)[v_n^{-1}].$$

We observe that the map $M(n-1) \to M(n-1)[v_n^{-1}]$ is injective. This implies that the map $\mathrm{MU}_* C_n S_{(p)} \to \mathrm{MU}_* C_{n-1} S_{(p)}$ is zero (thereby proving Theorem 1), and shows that we have a short exact sequence

$$0 \to M(n-1) \to M(n-1)[v_n^{-1}] \to \mathrm{MU}_* C_n S_{(p)} \to 0,$$

giving the isomorphism $\mathrm{MU}_* C_n S_{(p)} \simeq M(n)$. □

Proposition can also be used to get a bound on the discrepancy between $L_{E(n)}$ and the telescopic localization functor L_n^t.

Proposition 6. *Let X be any spectrum. Then the canonical map $L_n^t X \to L_{E(n)} X$ induces an isomorphism after smash product with* MU.

Proof. We work by induction on n. We wish to prove that the map

$$\mathrm{MU} \otimes L_n^t X \to \mathrm{MU} \otimes L_{E(n)} X \simeq L_{E(n)}(\mathrm{MU} \otimes X)$$

is an equivalence: that is, the map $\phi : \mathrm{MU} \otimes X \to \mathrm{MU} \otimes L_n^t X$ exhibits $\mathrm{MU} \otimes L_n^t X$ as an $E(n)$-localization of X. Since ϕ is obviously an $E(n)$-equivalence, it suffices to show that $\mathrm{MU} \otimes L_n^t X$ is $E(n)$-local. We have a cofiber sequence

$$Y \to X \to L_{n-1}^t X.$$

The inductive hypothesis implies that $L_n^t(\mathrm{MU} \otimes L_{n-1}^t X) \simeq \mathrm{MU} \otimes L_{n-1}^t X$ is $E(n-1)$-local, and therefore $E(n)$-local. It therefore suffices to show that $\mathrm{MU} \otimes L_n^t Y$ is $E(n)$-local By construction, Y is a direct limit of finite p-local spectra Y_α of type $\geq n$; since $L_{E(n)}$ is smashing, it suffices to show that each $\mathrm{MU} \otimes L_n^t Y_\alpha$ is MU-local. Since Y_α has type $\geq n$, $\mathrm{MU}_* Y_\alpha$ is an $I(n-1)$-torsion L-module, so that $\mathrm{MU}_* L_{E(n)} Y_\alpha \simeq (\mathrm{MU}_* Y_\alpha)[v_n^{-1}]$ by Proposition . On the other hand, we have seen that $L_n^t Y_\alpha \simeq Y_\alpha[f^{-1}]$, where f^{-1} is a v_n-self map of Y. To conclude that $\mathrm{MU}_* L_n^t Y_\alpha \simeq (\mathrm{MU}_* Y_\alpha)[v_n^{-1}]$, it suffices to prove the following:

Lemma 7. *Let Z be a finite p-local spectrum of type $\geq n$, and let $f : \Sigma^k Z \to Z$ be a v_n-self map of Z. Then, replacing f by a suitable power, we may assume that f induces the map v_n^i on $\mathrm{MU}_* Z$ (for some i).*

Proof. Let $R = Z \otimes DZ$, and regard f as an element of $\pi_* R$. Raising f to a suitable power, we may assume that $f \mapsto 0 \in K(m)_* R$ for $m \neq n$ and $f \mapsto v_n^i \in K(n)_* R$. We claim that $f^{p^k} = v_n^{ip^k}$ in $\mathrm{MU}_*(R)$ for $k \gg 0$. Since v_n^i and f commute in $\mathrm{MU}_* R$ (v_n being central) and the difference $v_n^i - f$ is p-power torsion (since $\pi_* R$ is p-power torsion), it suffices to show that $v_n^i - f$ is nilpotent. By the nilpotence theorem, it suffices to show that the image of $(v_n^i - f) \mapsto 0 \in K(m)_*(\mathrm{MU} \otimes R)$ for all m. This is clear for $m < n$ (since $\mathrm{MU} \otimes R$ is $K(m)$-acyclic). For $m \geq n$, we note that $v_n \in \pi_* \mathrm{MU}$ maps to 0 in $K(m)_* \mathrm{MU}$ for $m > n$ (since the formal group law of $K(m) \otimes \mathrm{MU}$ has height $> n$), so the statement holds since $f \mapsto 0 \in K(m)_* R$. In the case $m = n$, we are reduced to proving that the two images of v_n in $K(m)_* \mathrm{MU}$ coincide. This is clear: since $K(n) \otimes \mathrm{MU}$ is a cohomology theory with two complex orientations, the associated formal group laws (each of which has height $\geq n$) differ by a change of coordinates of the form $f(t) = t + b_1 t^2 + b_2 t^3 + \cdots$, so that the first nonvanishing coefficient of the p-series $[p](t)$ are the same. □

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