The Periodicity Theorem (Lecture 27)

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Let p be a prime number, fixed throughout this lecture. In the last lecture, we asserted that for every integer $n \geq 0$, there exists a finite p-local spectrum X of type n. If $n = 0$, this just means that the rational homology $\mathrm{H}_*(X; \mathbf{Q})$ is nonzero. We can achieve this by taking X to be the p-local sphere $S_{(p)}$.

When $n = 1$, we can define X to be the mod p Moore spectrum, which is defined by the cofiber sequence

$$S \xrightarrow{p} S \to X.$$

This has no rational homology. However, since multiplication by p annihilates $K(1)_*(S) \simeq \mathbf{F}_p[v_1^{\pm 1}]$, the map $K(1)_*(S) \to K(1)_*X$ is injective. In particular, $K(1)_*(X) \neq 0$, so that X has type 1.

For $n > 1$, it is somewhat harder to construct spectra of type n. We can try to mimic the previous construction. Namely, suppose that we are given a spectrum X of type n. We might try to find a self map $f : \Sigma^k X \to X$ so that we can form a cofiber sequence

$$\Sigma^k X \xrightarrow{f} X \to X/f,$$

and hope that X/f has type $n+1$. It is clear that X/f has type $\geq n$. To guarantee that X/f has type exactly $n+1$, we need to know two things:

(1) The $K(n)$-homology of X/f vanishes: in other words, f induces an isomorphism from $K(n)_*X$ to itself.

(2) The $K(n+1)$-homology of X/f does not vanish: that is, f does *not* induce an isomorphism from $K(n+1)_*X$ to itself.

This motivates the following definition:

Definition 1. Let X be a p-local finite spectrum, and let $n \geq 1$. A *v_n-self map* is a map $f : \Sigma^k X \to X$ with the following properties:

(a) f induces an isomorphism $K(n)_*X \to K(n)_*X$.

(b) For $m \neq n$, the induced map $K(m)_*X \to K(m)_*X$ is nilpotent.

Remark 2. If X is a p-local finite spectrum which admits a v_n-self map f, then X must have type $\geq n$. For if X has type $m < n$, then X/f has nonvanishing $K(m)$-homology (since f is not an isomorphism on $K(m)_*X$) but vanishing $K(n)$-homology.

Example 3. If X is a spectrum of type $> n$, then $K(n)_*X$ vanishes: it follows that the zero map $0 : X \to X$ is a v_n-self map.

The crucial case to consider is where X has type n. In this case, a v_n-self map $f : \Sigma^k X \to X$ will satisfy conditions (1) and (2) above, so that X/f will be a spectrum of type $n+1$. Consequently, to verify the existence of spectra of type n for every n, it will suffice to prove the following:

Theorem 4 (Periodicity Theorem)**.** *Let X be a finite p-local spectrum of type $\geq n$. Then X admits a v_n-self map.*

It will be useful to reformulate the notion of a v_n-self map. If X is a finite p-local spectrum, then $R = X \otimes DX$ has the structure of a ring spectrum. Moreover, giving a self map $\Sigma^k X \to X$ is equivalent to giving an element of $\pi_k R$. The condition of being a v_n-self map translates as follows:

Definition 5. Let R be a p-local ring spectrum. An element $x \in \pi_k R$ is a *v_n-element* if the image of x in $K(m)_* R$ is nilpotent for $m \neq n$, and invertible for $m = n$.

This is equivalent to saying that left multiplication by x induces a v_n-self map from R to itself. In particular, it implies that R has type $\geq n > 0$, so that the homotopy groups $\pi_* R$ consist of p-power torsion.

Lemma 6. *Let R be a finite p-local ring spectrum and let $x \in \pi_k R$ be a v_n-element. After raising x to a suitable power, we may assume that $x \mapsto v_n^a \in K(n)_*(R)$ and $x \mapsto 0 \in K(m)_* R$ for $m \neq n$.*

Proof. Recall that $K(m)_* R \simeq \mathrm{H}_*(R; \mathbf{F}_p)[v_m^{\pm 1}]$ for $m \gg 0$. It follows that x is nilpotent in $\mathrm{H}_*(R; \mathbf{F}_p)$. Replacing x by a suitable power, we may assume that $x \mapsto 0 \in \mathrm{H}_*(R; \mathbf{F}_p)$ and therefore $x \mapsto 0 \in K(m)_* R$ for $m \gg 0$. Consequently, there are only finitely many integers $m \neq n$ for which the image of x does not vanish in $K(m)_*(R)$. Each of these images is nilpotent; raising x to a power, we may assume that $x \mapsto 0 \in K(m)_* R$ for $m \neq n$.

Note that $K(n)_* R$ is a finite module over $\pi_* K(n) \simeq \mathbf{F}_p[v_n^{\pm 1}$. It follows that $(K(n)_* R)/(v_n - 1)$ is finite. The image of $x \in (K(n)_* R)/(v_n - 1)$ is a unit, so after raising x to a power we may assume that $x \mapsto 1 \in (K(n)_* R)/(v_n - 1) \simeq \bigoplus_{0 \leq i < 2(p^n - 1)} K(n)_i R$. It follows that $x \mapsto v_n^a \in K(n)_* R$ for some a. □

Lemma 7. *Let R be a $\mathbf{Z}_{(p)}$ algebra, and let $x, y \in R$ be commuting elements such that $x - y$ is torsion and nilpotent. Then $x^{p^k} = y^{p^k}$ for $k \gg 0$.*

Proof. We have

$$x^{p^k} = (y + (x - y))^{p^k} = y^{p^k} + \sum_{0 < i \leq p^k} \binom{p^k}{i} y^{p^k - i} (x - y)^i.$$

If $(x - y)^{p^a} = 0$, we can rewrite the right hand side as

$$y^{p^k} + \sum_{0 < i < p^a} \frac{p^k}{i} \binom{p^k - 1}{i - 1} y^{p^k - i} (x - y)^i.$$

Each expression $\frac{p^k}{i}$ is divisible by p^{k-a}, and therefore annihilates $x - y$ if $k \gg 0$. □

Lemma 8. *Let R be a finite p-local ring spectrum and let $x \in \pi_k R$ be a v_n-element. After raising x to a suitable power, we may assume that x is central in $\pi_* R$.*

Proof. Without loss of generality we may assume that x satisfies the conclusions of Lemma 6. Let $A = R \otimes DR$, and let $a, b \in \pi_k A$ be given by the self-maps of R given by left and right multiplication by x. Then a and b commute. Since A has type > 0, $\pi_* A$ is torsion, so $a - b \in \pi_k A$ is torsion. We claim that $a - b$ is nilpotent. To prove this, it suffices to show that the image of $a - b$ vanishes in $K(m)_* A$ for every integer m: in other words, the composite maps

$$R \to K(m) \otimes R \xrightarrow{x \times} K(m) \otimes R$$

$$R \to K(m) \otimes R \xrightarrow{\times x} K(m) \otimes R$$

agree. If $m \neq n$, this is clear (since $x \mapsto 0 \in K(m)_* R$. For $m = n$, we are reduced to proving that left and right multiplication by v_n^j induce the same self-map of $K(m) \otimes R$. This is clear, since $K(n)$ is a homotopy associative ring spectrum in the category of $\mathrm{MU}_{(p)}$-modules and v_n lies in the image of $\pi_* \mathrm{MU}_{(p)} \to \pi_* K(n)$.

Lemma 7 gives $a^{p^j} = b^{p^j}$ for $j \gg 0$. Replacing x by x^{p^j}, we can assume that $a = b$, so that left and right multiplication by x agree. □

Lemma 9. *Let R be a finite p-local ring spectrum and $x, y \in \pi_* R$ two v_n-elements. Then $x^a = y^b$ for suitable $a, b > 0$.*

Proof. Raising x and y to suitable powers, we may assume that $x, y \mapsto 0 \in K(m)_* R$ for $m \neq n$ and $x, y \mapsto v_n^j \in K(n)_* R$. Raising to a further power we may assume that x and y commute. Since R is of type > 0, $\pi_* R$ is torsion so that $x - y$ is a torsion element of $\pi_* R$. Then $x - y \mapsto 0 \in K(m)_* R$ for all m, so $x - y$ is nilpotent. Using Lemma 7 we conclude that $x^{p^j} = y^{p^j}$ for $j \gg 0$. □

Lemma 10. *Let X and Y be spectra which admit v_n-self maps $f : \Sigma^a X \to X$ and $g : \Sigma^b Y \to Y$. Let $h : X \to Y$ be any map. Then, replacing f and g by suitable powers, we may assume that $a = b$ and that the diagram*

$$\begin{array}{ccc} \Sigma^a X & \xrightarrow{h} & \Sigma^a Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{h} & Y \end{array}$$

commutes up to homotopy.

Proof. We can view the map h as given by $e : S \to DX \otimes Y$. The commutativity of the diagram then amounts to a homotopy $(Df \otimes \mathrm{id}_Y) \circ e \simeq (\mathrm{id}_{DX} \otimes g) \circ e$. Since $Df \otimes \mathrm{id}_Y$ and $\mathrm{id}_{DX} \otimes g$ are two v_n-self maps of $DX \otimes Y$, this identity will hold after replacing f and g by appropriate powers (Lemma 9). □

Proposition 11. *Let $\mathcal{T}$ be the collection of p-local finite spectra which admit a v_n-self map. Then $\mathcal{T}$ is thick.*

Proof. It is clear that $0 \in \mathcal{T}$ and that $\mathcal{T}$ is closed under suspension and desuspension. We next show that $\mathcal{T}$ is closed under taking cofibers. Let $h : X \to Y$ be a map of p-local finite spectra which admit v_n-self maps f and g. By virtue of Lemma 10, we may assume that the diagram

$$\begin{array}{ccc} \Sigma^k X & \xrightarrow{h} & \Sigma^k Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{h} & Y \end{array}$$

commutes up to homotopy. We may further assume that f and g are zero on $K(m)$-homology for $m \neq n$.

A choice of homotopy induces a map of cofibers $x : \Sigma^k(Y/X) \to Y/X$. We claim that x is a v_n-self map. Since f and g induce an isomorphism on $K(n)$-homology, the associated long exact sequence in homology shows that x induces an isomorphism on $K(n)$-homology. For $m \neq n$, we have a map of exact sequences

$$\begin{array}{ccccc} K(m)_{*-k} Y & \longrightarrow & K(m)_{*-k}(Y/X) & \longrightarrow & K(m)_{*-k-1} X \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle 0} \\ K(m)_* Y & \xrightarrow{\phi} & K(m)_* Y/X & \longrightarrow & K(m)_{*-1} X. \end{array}$$

It follows that x carries $K(m)_{*-k}(Y/X)$ into the image of ϕ and that multiplication by x is trivial on the image of ϕ, so that x^2 is trivial on $K(m)_* Y/X$.

It remains to prove that $\mathcal{T}$ is stable under retracts. Let X and Y be p-local spectra and assume that $X \oplus Y$ admits a v_n-self map f. Raising f to a power, we may assume that f vanishes on $K(m)_*(X \oplus Y)$ for $m \neq n$ and is given by multiplication by v_n^j on $K(n)_*(X \oplus Y)$. Then the composite map

$$\Sigma^k X \to \Sigma^k (X \oplus Y) \xrightarrow{f} X \oplus Y \to X$$

has the same properties, and is therefore a v_n self map. □

Let $\mathcal{T}$ be the thick subcategory of Proposition 11. The periodicity theorem can be restated as follows: $\mathcal{T}$ contains every spectrum of type $\geq n$. By the thick subcategory theorem, this is equivalent to the following result, which we assert without proof:

Proposition 12. *For every integer $n \geq 0$, there exists a finite p-local spectrum X of type n, and a v_n-self map $f : \Sigma^k X \to X$.*