Thick Subcategories (Lecture 26)

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Let p be a prime number, fixed throughout this lecture.

Let $\mathcal{C}$ be a full subcategory of the category of p-local spectra which is stable under homotopy colimits and desuspension, and which is generated under homotopy colimits by a small subcategory. The theory of Bousfield localization allows us to associate to every p-local spectrum X a canonical fiber sequence

$$C(X) \to X \to L(X),$$

where $C(X) \in \mathcal{C}$ and $L(X)$ is $\mathcal{C}$-local (that is, every map from an object of $\mathcal{C}$ into $L(X)$ is nullhomotopic).

Let $\mathcal{C}_0$ be the collection of all finite p-local spectra contained in $\mathcal{C}$. If $\mathcal{C}_0$ generates $\mathcal{C}$ under homotopy colimits, then the localization functor L is smashing. In this case, $\mathcal{C}_0$ determines $\mathcal{C}$ and vice versa. The following definition axiomatizes the expected properties of $\mathcal{C}_0$:

Definition 1. Let $\mathcal{T}$ be a full subcategory of the homotopy category of finite p-local spectra. We say that $\mathcal{T}$ is *thick* if it contains 0, is closed under the formation of fibers and cofibers, and if every retract of a spectrum belonging to $\mathcal{T}$ also belongs to $\mathcal{T}$.

Remark 2. Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. If $X \in \mathcal{T}$ and Y is any finite p-local spectrum, then $X \otimes Y \in \mathcal{T}$. Indeed, the collection of p-local finite spectra Y which for which $X \otimes Y \in \mathcal{T}$ is itself thick. Since it contains the p-local sphere $S_{(p)}$, it contains all finite p-local spectra (every finite p-local spectrum admits a finite cell decomposition).

Remark 3. Let $\mathcal{T}$ be any thick subcategory of the category of finite p-local spectra, and let $\mathcal{C}$ be the collection of p-local spectra generated by $\mathcal{T}$ under homotopy colimits. Every object $X \in \mathcal{C}$ can be written as a filtered colimit of objects $X_\alpha \in \mathcal{T}$. In particular, if X is a finite p-local spectrum, then the identity map $X \to \varinjlim X_\alpha$ factors through some X_α. Thus X is a retract of X_α and so $X \in \mathcal{T}$. Consequently, the construction $\mathcal{T} \mapsto \mathcal{C}$ determines a bijection between thick subcategories of finite p-local spectra and subcategories $\mathcal{C}$ of the category of all p-local spectra, which are stable under desuspension and generated by p-local finite spectra under homotopy colimits.

Our next goal is to describe some thick subcategories. We begin with the following observation:

Lemma 4. *Let X be a finite p-local spectrum. Suppose that $K(n)_*(X) \simeq 0$ for some $n > 0$. Then $K(n-1)_*(X) \simeq 0$.*

To prove this, we let R denote the ring spectrum obtained by smashing $\mathrm{MU}_{(p)}[v_n^{-1}]$ over $\mathrm{MU}_{(p)}$ with the spectra $\{M(k)\}_{k \neq p^n-1, p^{n-1}-1}$. For simplicity, let us assume $n > 1$ (the proof in the case $n = 1$ is essentially the same, but the notation changes). Then R is a ring spectrum with $\pi_* R \simeq \mathbf{F}_p[v_{n-1}, v_n^{\pm 1}]$. In particular, $\pi_0 R$ is equivalent to the polynomial ring $\mathbf{F}_p[v_{n-1}^a v_n^{-b}] = \mathbf{F}_p[t]$ where (a,b) is the minimal solution to $a(p^{n-1}-1) - b(p^n - 1) = 0$. Note that for every integer k, $R_k(X)$ is a finitely generated module over $\pi_0 R$. We have a cofiber sequence

$$\Sigma^{2(p^{n-1}-1)} R \stackrel{v_{n-1}}{\to} R \to K(n).$$

Since $K(n)_* X \simeq 0$, we conclude that multiplication by v_{n-1} and hence multiplication by t acts invertibly on each $R_k(X)$. It follows that each $R_k(X)$ is a torsion module over $\mathbf{F}_p[t]$, and is therefore annihilated by almost every irreducible polynomial in $\mathbf{F}_p[t]$. In particular, we can choose a nonzero polynomial $f(t)$ which annihilates each $R_k(X)$ for $0 \leq k < 2(p^n - 1)$ and therefore for all values if k (since $\pi_* R$ is periodic with period $2(p^n-1)$). Without loss of generality, $f(t)$ is divisible by t. For $k \gg 0$, the product $f(t)v_n^k$ can be written as a polynomial in v_{n-1} and v_n, and therefore comes from $\pi_* \mathrm{MU}$. We can therefore localize R to obtain a new ring spectrum $R[f(t)^{-1}]$ with $R[f(t)^{-1}]_* X \simeq R_* X[f(t)^{-1}] \simeq 0$.

By construction, $R[f(t)^{-1}]$ has a complex orientation and the associated formal group has height exactly $n-1$ (since $f(t)$ is divisible by t, so v_{n-1} is invertible in $\pi_* R[f(t)^{-1}]$). It follows that $R[f(t)^{-1}] \otimes K(m)$ vanishes for $m \neq n-1$. Since $R[f(t)^{-1}] \neq 0$, $R[f(t)^{-1}] \otimes K(n-1) \neq 0$ and therefore contains $K(n-1)$ as a retract. Since $X \otimes R[f(t)^{-1}] \simeq 0$, we conclude that $X \otimes R[f(t)^{-1}] \otimes K(n-1) \simeq 0$ so that $X \otimes K(n-1) \simeq 0$, as desired.

Remark 5. Let X be a finite p-local spectrum. Then $\mathrm{H}_*(X; \mathbf{F}_p) \simeq 0$ if and only if $X \simeq 0$. Moreover, $\mathrm{H}_k(X; \mathbf{F}_p)$ vanishes for almost all values of k. For $n \gg 0$, the Atiyah-Hirzebruch spectral sequence for $K(n)_*(X)$ degenerates to give $K(n)_*(X) \simeq \mathrm{H}_*(X; \mathbf{F}_p)[v_n^{\pm 1}]$. It follows that if $X \neq 0$, then $K(n)_*(X) \neq 0$ for $n \gg 0$.

Definition 6. We say that a p-local finite spectrum X has *type* n if $K(n)_*(X) \neq 0$ but $K(m)_*(X) \simeq 0$ for $m < n$. For example, X has *type* 0 if $\mathrm{H}_*(X; \mathbf{Q}) \simeq 0$, or equivalently if $\mathrm{H}_*(X; \mathbf{Z})$ is not a torsion group.

Every nonzero finite p-local spectrum X has type n for some unique n. By convention, we will say that the spectrum 0 has type ∞.

Definition 7. Let $\mathcal{C}_{\geq n}$ be the collection of finite p-local spectra which have type $\geq n$. In other words, $X \in \mathcal{C}_{\geq n}$ if and only if $K(m)_*(X) \simeq 0$ for $m < n$.

Using the long exact sequence in $K(m)$-homology, we see that if we are given a cofiber sequence

$$X' \to X \to X'',$$

and any two of X', X, and X'' has type $\geq n$, then so does the third. Moreover, it is clear that any retract of a spectrum of type $\geq n$ is also of type $\geq n$. Consequently, $\mathcal{C}_{\geq n}$ is a thick subcategory of the category of finite p-local spectra.

The main result of this lecture is the following:

Theorem 8 (Thick Subcategory Theorem)**.** *Let $\mathcal{T}$ be a thick subcategory of finite p-local spectra. Then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $0 \leq n \leq \infty$.*

In other words, the $\mathcal{C}_{\geq n}$ are exactly the thick subcategories of finite p-local spectra.

Remark 9. It is not yet clear that the classes $\mathcal{C}_{\geq n}$ are different for distinct n. This is equivalent to the following assertion: for every nonnegative integer n, there exists a finite p-local spectrum of type n. We will discuss the proof of this theorem in the next lecture.

Let $\mathcal{T}$ be as in Theorem 8. If $\mathcal{T}$ contains only the zero spectrum, then we can take $n = \infty$. Otherwise, there exists a nonzero spectrum $X \in \mathcal{T}$ having type n for $n < \infty$. Choose X so that n is minimal; we wish to prove that $\mathcal{T} = \mathcal{C}_{\geq n}$. The inclusion $\mathcal{T} \subseteq \mathcal{C}_{\geq n}$ is clear (otherwise, $\mathcal{T}$ would contain a spectrum of type $< n$, contradicting minimality). Theorem 8 can therefore be reformulated as follows:

Proposition 10. *Let $\mathcal{T}$ be a thick subcategory containing a type n spectrum X. If Y is a spectrum of type $\geq n$, then $Y \in \mathcal{T}$.*

To prove this, let DX denote the (p-local) Spanier-Whitehead dual of X. The identity map $X \to X$ is classified by a map $e : S_{(p)} \to X \otimes DX$. Since X has type n, we note that e induces an injection $K(m)_*(S_{(p)}) \to K(m)_*(X \otimes DX) \simeq K(m)_*(X) \otimes_{\mathbf{F}_p[v_m^{\pm 1}]} K(m)_*(X)^\vee$ for $m \geq n$. Form a fiber sequence

$$F \xrightarrow{f} S_{(p)} \to X \otimes DX.$$

It follows that the map $K(m)_*F \to K(m)_*(S_{(p)})$ is zero for $m \geq n$. Consider the composite map

$$g : F \xrightarrow{f} S_{(p)} \to Y \otimes DY.$$

Then g induces the zero map $K(m)_*F \to K(m)_*(Y \otimes DY)$ for $m \geq n$ (since f has the same property) and also for $m < n$ (since Y has type $\geq n$, so that $K(m)_*(Y \otimes DY) \simeq 0$). By the nilpotence theorem, we conclude that some smash power $F^{\otimes k} \to (Y \otimes DY)^{\otimes k}$ is nullhomotopic. Composing with the multiplication on $Y \otimes DY$, we get a nullhomotopic map $F^{\otimes k} \to Y \otimes DY$, which corresponds to the composition

$$F^{\otimes k} \otimes Y \xrightarrow{f} F^{\otimes k-1} \otimes Y \xrightarrow{f} \cdots \to Y.$$

It follows that Y is a retract of the cofiber $Y/(F^{\otimes k} \otimes Y)$. Consequently, to show that $Y \in \mathcal{T}$, it will suffice to show that $Y/(F^{\otimes k} \otimes Y) \in \mathcal{T}$.

The spectrum $Y/F^{\otimes k} \otimes Y$ admits a finite filtration by spectra of the form $(F^{\otimes a} \otimes Y)/(F^{\otimes a+1} \otimes Y)$. Since $\mathcal{T}$ is thick, it will suffice to show that each of these belongs to $\mathcal{T}$. Each of these spectra has the form

$$F^{\otimes a} \otimes Y \otimes (S_{(p)}/F) \simeq F^{\otimes a} \otimes Y \otimes DX \otimes X,$$

and therefore belongs to $\mathcal{T}$ since $X \in \mathcal{T}$ (Remark 2).