Math 261y: von Neumann Algebras (Lecture 10)

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The following result provides an intrinsic characterization of von Neumann algebras:

Theorem 1. *Let A be a C^*-algebra. Suppose there exists a Banach space E and a Banach space isomorphism $A \simeq E^\vee$. Then there exists a von Neumann algebra B and an isomorphism of C^*-algebras $A \to B$ (in other words, A admits the structure of a von Neumann algebra).*

We will prove Theorem 1 under the following additional assumption:

($*$) For each $a \in A$, the operations $l_a, r_a : A \to A$ given by left multiplication on A are continuous with respect to the weak $*$-topology (arising from the identification $A \simeq E^\vee$).

Remark 2. We have seen that every von Neumann algebra admits a Banach space predual, such that the weak $*$-topology coincides with the ultraweak topology. Since multiplication in a von Neumann algebra is separately continuous in each variable for the ultraweak topology, condition ($*$) is satisfied in any von Neumann algebra.

Let us now explain the proof of Theorem 1. Fix an isomorphism $\phi : A \to E^\vee$. We can think of ϕ as giving a bilinear pairing between A and E, which in turn determines a bounded operator $\phi' : E \to A^\vee$. Let $\hat{\phi} : A^{\vee\vee} \to E^\vee$ denote the dual of ϕ'. The map $\hat{\phi}$ is continuous with respect to the weak $*$-topologies on $A^{\vee\vee}$ and $M^\vee$, respectively, and fits into a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & M^\vee \\ & {\scriptstyle\rho}\searrow \quad \nearrow{\scriptstyle\hat{\phi}} & \\ & A^{\vee\vee}. & \end{array}$$

Here ρ is the canonical map from A into its double dual. The map $\hat{\phi}$ is uniquely determined by these properties (since A is dense in $A^{\vee\vee}$ with respect to the weak $*$-topology). We have seen that $A^{\vee\vee}$ admits the structure of a von Neumann algebra, and that ρ can be considered as a C^*-algebra homomorphism which exhibits $A^{\vee\vee}$ as the von Neumann algebra envelope of A. Let $r = \phi^{-1} \circ \hat{\phi}$. Then r is a left inverse to the canonical inclusion $\rho : A \to A^{\vee\vee}$.

Fix an element $a \in A$. Let $l_a : A \to A$ denote the operation given by left multiplication by A, and let $l_{\rho(a)} : A^{\vee\vee} \to A^{\vee\vee}$ be defined similarly. Consider the diagram

$$\begin{array}{ccc} A^{\vee\vee} & \xrightarrow{r} & A \\ \downarrow{\scriptstyle l_{\rho(a)}} & & \downarrow{\scriptstyle l_a} \\ A^{\vee\vee} & \xrightarrow{r} & A. \end{array}$$

Since ρ is an algebra homomorphism, this diagram commutes on the subset $\rho(A) \subseteq A^{\vee\vee}$. Using assumption ($*$), we see that all of the maps in this diagram are continuous if we regard $A^{\vee\vee}$ and $A \simeq E^\vee$ as equipped with the weak $*$-topologies. Since the image of ρ is weak $*$-dense in $A^{\vee\vee}$, we conclude that the diagram commutes.

Let $K \subseteq A^{\vee\vee}$ denote the kernel of r. Since r is continuous with respect to the weak $*$-topologies, K is closed with respect to the weak $*$-topology on $A^{\vee\vee}$ (which coincides with the ultraweak topology). If $x \in K$, we have

$$r(\rho(a)x) = ar(x) = 0$$

so that $\rho(a)x = K$. The set $\{y \in A^{\vee\vee} : yx \in K\}$ is ultraweakly closed (since multiplication by x is ultraweakly continuous) and contains the image of ρ. Since $\rho(A)$ is ultraweakly dense in $A^{\vee\vee}$, we deduce that $\{y \in A^{\vee\vee} : yx \in K\}$ contains all of $A^{\vee\vee}$. It follows that K is a left ideal in $A^{\vee\vee}$.

The same argument shows that K is a right ideal in $A^{\vee\vee}$. Since K is ultraweakly closed, the results of the last lecture show that K is a $*$-ideal, and that the von Neumann algebra $A^{\vee\vee}$ decomposes as a product

$$A^{\vee\vee} \simeq A^{\vee\vee}/K \times K.$$

Set $B = A^{\vee\vee}/K$. The composite map

$$A \to A^{\vee\vee} \to B$$

is a C^*-algebra homomorphism and an isomorphism on the level of vector spaces, hence an isomorphism of C^*-algebras. This completes the proof of Theorem 1 (under the additional assumption $(*)$).

In fact, we can say a bit more. Let us regard E as a subspace of its double dual $E^{\vee\vee} \simeq A^{\vee}$, so that every vector e determines a functional $\mu_e : A \to \mathbf{C}$. Every such functional extends to a weak $*$-continuous map $\hat{\mu}_e : A^{\vee\vee} \to \mathbf{C}$, given by the composition

$$A^{\vee\vee} \stackrel{\hat{\phi}}{\to} E^{\vee} \stackrel{e}{\to} \mathbf{C}.$$

This composite map is ultraweakly continuous (since the weak $*$-topology on $A^{\vee\vee}$ coincides with the ultraweak topology) and annihilates K (since $K = \ker(r) = \ker(\hat{\phi})$). It follows that $\hat{\mu}_e$ descends to an ultraweakly continuous functional $B \to \mathbf{C}$. In other words, the functional μ_e is ultraweakly continuous if we regard A as a von Neumann algebra using the isomorphism $A \simeq B$.

Let $F \subseteq A^{\vee}$ be the collection of ultraweakly continuous functionals with respect to our von Neumann algebra structure on A, so that we can regard E as a closed subspace of F. Consider the composite map

$$A \to A^{\vee\vee} \to F^{\vee} \to E^{\vee}.$$

Since A is a von Neumann algebra, the composition of the first two maps is an isomorphism. Since the composition of all three maps is an isomorphism by assumption, we conclude that the map $F^{\vee} \to E^{\vee}$ is an isomorphism. This implies that $E = F$: that is, E can be identified with the subspace of $A^{\vee}$ consisting of *all* ultraweakly continuous functionals on A. In particular, the weak $*$-topology on A agrees with the ultraweak topology given by the von Neumann algebra structure on A.

It is natural to ask to what extent the Banach space E appearing in Theorem 1 is unique. Suppose we are given two Banach spaces E and E', together with isomorphisms

$$E^{\vee} \simeq A \simeq E'^{\vee}.$$

Can we then identify E with E'? In this situation, we can think of E and E' as subspaces of the dual space $A^{\vee}$; we then ask: do these subspaces necessarily coincide? Our analysis shows that E determines an isomorphism of A with a von Neumann algebra B, and that as a subspace of $A^{\vee}$ we can identify E with those linear functionals which are ultraweakly continuous on B. Similarly, E' determines an isomorphism $A \simeq B'$. Asking if $E = E'$ (as subspaces of $A^{\vee}$) is equivalent to asking if the C^*-algebra isomorphism $B \simeq A \simeq B'$ carries ultraweakly continuous functionals on B to ultraweakly continuous functionals on B'. We can therefore phrase the question as follows:

Question 3. Let B and B' be von Neumann algebras, and let $f : B \to B'$ be a $*$-algebra isomorphism. Is f necessarily an isomorphism of von Neumann algebras? That is, is f automatically continuous with respect to the ultraweak topologies?

We will answer this question in the affirmative. Equivalently, we will show that the isomorphism f carries ultraweakly continuous functionals μ on B' to ultraweakly continuous functionals on B. We first note that it suffices to treat the case of *positive* functionals:

Lemma 4. *Let $B \subseteq B(V)$ be a von Neumann algebra. Then the vector space of ultraweakly continuous functionals on B is generated by ultraweakly continuous states.*

Proof. Every ultraweakly continuous functional $\mu : B \to \mathbf{C}$ is given by

$$\mu(x) = \sum (x(v_i), w_i)$$

for some sequences of vectors $v_i, w_i \in V$ with $\sum ||v_i||^2 < \infty, \sum ||w_i||^2 < \infty$. Replacing V by $V^{\oplus\infty}$, we may assume that μ is given by $\mu(x) = (x(v), w)$. Then

$$\mu(x) = \frac{1}{4}(x(v+w), v+w) + \frac{i}{4}(x(v+iw), v+iw) + -\frac{1}{4}(x(v-w), v-w) - \frac{i}{4}(x(v-iw), v-iw)$$

is a linear combination of ultraweakly continuous positive functionals, each of which is a multiple of an ultraweakly continuous state. □

Definition 5. Let B be a von Neumann algebra. We say that an element $e \in B$ is a *projection* if e is Hermitian and $e^2 = e$. Given a pair of projections e and e', we will write $e \leq e'$ if $ee' = e'e = e$. We say that e and e' are *orthogonal* if $ee' = e'e = 0$. In this case, $e + e'$ is also a projection, satisfying

$$e \leq e + e' \geq e'.$$

If B is given as the set of bounded operators on some Hilbert space V, then an element $e \in B$ is a projection if and only if it is given by orthogonal projection onto some closed subspace $W \subseteq V$. Let us denote such a projection by e_W. Note that $e_W \leq e_{W'}$ if and only if $W \subseteq W'$, and that e_W and $e_{W'}$ are orthogonal if and only if W and W' are orthogonal.

Suppose we are given a collection of mutually orthogonal projections $\{e_{W_\alpha}\}_{\alpha \in I}$ in B. Let W be the closed subspace of V generated by the subspaces W_α. Then the collection of all finite sums $\sum_{\alpha \in I_0} e_{W_\alpha}$ converges to the projection e_W in the ultraweak topology (in fact, it even converges in the ultrastrong topology). It follows that $e_W \in B$. We can characterize e_W as the smallest projection satisfying $e_W \geq e_{W_\alpha}$ for every index α.

We say that a state $\mu : B \to \mathbf{C}$ is *completely additive* if, for every collection of mutually orthogonal projections $\{e_\alpha\}$ in B, we have

$$\mu(\sum_\alpha e_\alpha) = \sum_\alpha \mu(e_\alpha).$$

It is clear from the definition that every ultraweakly continuous state on a von Neumann algebra B is completely additive. In the next lecture, we will prove the converse:

Proposition 6. *Let B be a von Neumann algebra and let $\mu : B \to \mathbf{C}$ be a completely additive state. Then μ is ultraweakly continuous.*

It is clear that any C^*-algebra isomorphism $B \simeq B'$ carries completely additive states to completely additive states. Using Proposition 6, we deduce that such an isomorphism is ultraweakly continuous. In fact, we get the following more general result:

Corollary 7. *Let $f : B \to B'$ be a C^*-algebra homomorphism between von Neumann algebras. Then f is a von Neumann algebra homomorphism (that is, f is ultraweakly continuous) if and only if it satisfies the following condition:*

- $(*)$ *For every collection of mutually orthogonal projections $\{e_\alpha\}$ in B having sum e, we have $f(e) = \sum_\alpha f(e_\alpha)$.*

In particular, if f is an isomorphism of C^*-algebras, then f is ultraweakly continuous.