Lecture 9: Divisors

October 22, 2018

Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Let Y denote the set of all isomorphism classes of characteristic zero untilts $K = (K, \iota)$ of $C^\flat$. For each $0 < a \leq b < 1$, we let $Y_{[a,b]} \subseteq Y$ denote the subset consisting of those untilts K satisfying $a \leq |p|_K \leq b$.

Recall that our heuristic is that Y behaves somewhat like a Riemann surface, with the ring $B_{[a,b]}$ (obtained by completing $A_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the Gauss norms $|\bullet|_a$ and $|\bullet|_b$) behaves like the ring of holomorphic functions on $B_{[a,b]}$.

For each characteristic zero untilt K of $C^\flat$, we let $B_{\text{dR}}^+(K)$ denote the discrete valuation ring constructed in the previous lecture (with residue field K). In Lecture 8, we proved that if $a \leq |p|_K \leq b$. then the canonical map $\theta : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_K$ lifts to a map $e : B_{[a,b]} \rightarrow B_{\text{dR}}^+(K)$. For each $x \in B_{[a,b]}$, we let $\text{ord}_K(x) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ denote the valuation of $e(x)$ in $B_{\text{dR}}^+(K)$ (so that $\text{ord}_K(x) = \infty$ if $e(x) = 0$, and otherwise $\text{ord}_K(x)$ is the unique integer n such that $(e(x))$ coincides with the nth power of the maximal ideal of $B_{\text{dR}}^+(K)$. We will refer to $\text{ord}_K(x)$ as the *order of vanishing* of x at the untilt K.

Our main objective over the next several lectures will be to prove the following:

Theorem 1. *Assume that $C^\flat$ is algebraically closed, and fix $0 < a \leq b < 1$. Then:*

(1) *Let x be a nonzero element of $B_{[a,b]}$. Then $\text{ord}_K(x) < \infty$ for each $K \in Y_{[a,b]}$. Moreover, there are only finitely many elements $K \in Y_{[a,b]}$ for which $\text{ord}_K(x) \neq 0$.*

The first part of (1) asserts that each of the maps $e : B_{[a,b]} \rightarrow B_{\text{dR}}^+(K)$ is injective. In particular, the ring $B_{[a,b]}$ is an integral domain.

(2) *Let x and y be nonzero elements of $B_{[a,b]}$. Then x is divisible by y if and only if $\text{ord}_K(x) \geq \text{ord}_K(y)$ for each $K \in Y_{[a,b]}$.*

For each nonzero element $x \in B_{[a,b]}$, we let $\text{Div}_{[a,b]}(x)$ denote the formal sum $\sum_{K \in Y_{[a,b]}} \text{ord}_K(x) \cdot K$, which we regard as an element of the free abelian group generated by the set $Y_{[a,b]}$. If x is a nonzero element of B, we let $\text{Div}(x)$ denote the formal sum $\sum_{K \in Y} \text{ord}_K(x) \cdot K$. Beware that this latter sum may have infinitely many terms; however, it has only finitely many summands lying in each $Y_{[a,b]}$.

Example 2. Let ξ be a distinguished element of $\mathbf{A}_{\text{inf}}$. We distinguish two cases:

(1) If ξ is a unit multiple of p, then it is invertible in the ring B. In this case, we have $\text{Div}(\xi) = 0$.

(2) If ξ is not a unit multiple of p, then there is a unique characteristic zero untilt K of $C^\flat$ such that the map $\theta : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_K$ annihilates ξ. By construction, the image of ξ is a uniformizer of the discrete valuation ring B_{dR}^+. It follows that $\text{ord}_K(\xi) = 1$ and $\text{ord}_{K'}(\xi) = 0$ for $K' \neq K$, so the divisor $\text{Div}(\xi)$ is equal to K.

Example 3. Let x be an element of $C^\flat$ satisfying $0 < |x-1|_{C^\flat} < 1$. We saw in the previous lecture that there is precisely one Frobenius orbit of untilts on which $\log([x])$ vanishes. Moreover, one of the untilts K belonging to this locus is given by the vanishing locus of the distinguished element

$$\xi = 1 + [x^{1/p}] + \cdots + [x^{(p-1)/p}] = \frac{[x]-1}{[x^{1/p}]-1} \in \mathbf{A}_{\text{inf}}.$$

Note that the image of $[x^{1/p}]$ in K is a primitive pth root of unity ζ_p, so that $\zeta_p - 1$ is invertible in K (though not in $\mathcal{O}_K$) and therefore $[x^{1/p}] - 1$ is invertible in $B^+_{\text{dR}}(K)$. It follows that $[x] - 1$ is a unit multiple of ξ in B^+_{dR}, and is therefore a uniformizer. The congruence

$$\log([x]) = \sum_{k>0} \frac{(-1)^{k+1}}{k}([x]-1)^k \equiv [x]-1 \pmod{([x]-1)^2}$$

shows that $\text{ord}_K(\log([x])) = 1$. By symmetry, we conclude that $\log([x])$ vanishes to order exactly one at each point belonging to the Frobenius orbit of K: that is, we have

$$\text{Div}(\log([x])) = \sum_{n\in\mathbf{Z}} \varphi^n(K).$$

For nonzero elements $x, y \in B$, we write $\text{Div}(x) \geq \text{Div}(y)$ if $\text{ord}_K(x) \geq \text{ord}_K(y)$ for each $K \in Y$. From Theorem 1, we immediately deduce the following:

Corollary 4. *Assume that $C^\flat$ is algebraically closed. The ring B is an integral domain. Moreover, if x and y are nonzero elements of B, then x is divisible by y if and only if $\text{Div}(x) \geq \text{Div}(y)$.*

We now state two further results we will prove later:

Theorem 5. *The canonical map $\mathbf{Q}_p \to B^{\varphi=1}$ is an isomorphism, and the vector space $B^{\varphi=p^n}$ vanishes when n is negative.*

Theorem 6. *Suppose that $C^\flat$ is algebraically closed. Then every untilt of $C^\flat$ is algebraically closed.*

Remark 7. The rough idea of Theorem 6 is easy to explain. If $C^\flat$ admits an untilt K which is not algebraically closed, then K admits some finite algebraic extension L. In this case, one would like to argue that L is also a perfectoid field and that $L^\flat$ is a finite algebraic extension of $K^\flat \simeq C^\flat$, contradicting our assumption that $C^\flat$ is algebraically closed. Fleshing out this argument requires some work, which we defer to a future lecture.

Let us collect some consequences.

Corollary 8. *Assume that $C^\flat$ is algebraically closed. Then every untilt K of $C^\flat$ belongs to the vanishing locus of $\log([x])$, for some element $x \in C^\flat$ satisfying $0 < |x-1|_{C^\flat} < 1$. Moreover, the map*

$$\psi : \{y \in C^\flat : |y-1|_{C^\flat} < 1\} \xrightarrow{\log(y^\sharp)} K$$

is surjective, whose kernel is generated by x (as a subspace of the $\mathbf{Q}_p$-vector space $\{y \in C^\flat : |y-1|_{C^\flat} < 1\}$).

Proof. To prove the first assertion, it will suffice (by results of the previous lecture) to show that every untilt K of $C^\flat$ contains a compatible system of p^nth roots of unity; this is immediate from Theorem 6. Note that if z is an element of K satisfying $|z|_K < |p|_K^{1/(p-1)}$, then the exponential $\exp(z)$ is well-defined. Since K is algebraically closed (Theorem 6), we can choose a compatible system of p^nth roots of unity of $\exp(z)$: that is, we can write $\exp(z) = y^\sharp$ for some $y \in C^\flat$. Some simple estimates yield $|y-1|_{C^\flat} = |\exp(z)-1|_K = |z|_K < 1$, so that $z = \log(y^\sharp)$. It follows that the image of ρ contains all *sufficiently small* elements of K. However, ρ is a map of $\mathbf{Q}_p$-vector spaces, and every element of K becomes sufficiently small after multiplying by a large power of p. This proves that ρ is surjective, and the kernel of ρ was described in the previous lecture. □

Corollary 9. *Assume that $C^\flat$ is algebraically closed. Then the map*

$$1 + \mathfrak{m}_C^\flat \xrightarrow{\log([x])} B^{\varphi = p}$$

is an isomorphism.

Proof. Injectivity is clear (since $\log([x])$ vanishes on a single Frobenius orbit of Y for $x \neq 1$). To prove surjectivity, we must show that every element $f \in B^{\varphi=p}$ has the form $\log([x])$ for some $x \in \mathfrak{m}_C^\flat + 1$. Assume that $f \neq 0$ (otherwise, we can take $x = 1$). We first claim that the divisor $\mathrm{Div}(f)$ is nonzero. Otherwise, Corollary 4 would imply that f is invertible. Then the inclusion $f \in B^{\varphi=p}$ guarantees $f^{-1} \in B^{\varphi=p^{-1}}$, so that $f^{-1} = 0$ by Theorem 5, which is clearly impossible.

Since $\mathrm{Div}(f)$ is nonzero, we can choose an untilt K of $C^\flat$ satisfying $\mathrm{ord}_K(f) \geq 1$. Since f belongs to $B^{\varphi=p}$, it follows that $\mathrm{ord}_{K'}(f) \geq 1$ for any $K' \in Y$ which belongs to the Frobenius orbit of K. Choose an element $x \in 1 + \mathfrak{m}_C^\flat$ such that $x \neq 1$ and $\log([x])$ vanishes at K. Then

$$\mathrm{Div}(\log([x])) = \sum_{n \in \mathbf{Z}} \varphi^n(K) \leq \mathrm{Div}(f).$$

Applying Corollary 4, we deduce that f is divisible by $\log([x])$: that is, we can write $f = \log([x]) \cdot g$. Since both f and $\log([x])$ belong to $B^{\varphi=p}$ (and B is an integral domain), it follows that g belongs to $B^{\varphi=1}$. Using Theorem 5, we conclude that $g \in \mathbf{Q}_p$ is a scalar. We can therefore arrange (after replacing x by a suitable scalar multiple in the $\mathbf{Q}_p$-vector space $1 + \mathfrak{m}_C^\flat$) that $g = 1$, so that $f = \log([x])$ as desired. $\square$