Lecture 8: The Field B_{dR}

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Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Fix an element $\pi \in C^\flat$ with $0 < |\pi|_{C^\flat} < 1$, and let B denote the completion of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the family of Gauss norms $|\bullet|_\rho$ for $0 < \rho < 1$. Recall our heuristic picture: B is an analogue of the ring of holomorphic functions on the punctured unit disk $D^\times = \{z \in \mathbf{C} : 0 < |z| < 1\}$. In the previous lecture, we studied some elements of the eigenspace $B^{\varphi = p}$ which can be described in two equivalent ways:

- As convergent Teichmüller expansions $\sum \frac{[a^{p^n}]}{p^n}$, where $|a|_{C^\flat} < 1$.
- As logarithms $\log([x])$, where $|x - 1|_{C^\flat} < 1$.

We now study the "vanishing loci" of these objects, viewed as functions on the "space" Y of (characteristic zero) untilts $y = (K, \iota)$ of $C^\flat$."

Construction 1. Let $\mathbf{Q}_p^{\mathrm{cyc}}$ denote the perfectoid field introduced in Lecture 2 (given by the completion of the union of cyclotomic extensions $\bigcup_{n>0} \mathbf{Q}_p[\zeta_{p^n}]$). We let ϵ denote the sequence of element of $\mathbf{Q}_p^{\mathrm{cyc}}$ given by $(1, \zeta_p, \zeta_{p^2}, \zeta_{p^3}, \cdots)$, which we regard as an element of the tilt $(\mathbf{Q}_p^{\mathrm{cyc}})^\flat$. By construction, we have $\epsilon^\sharp = 1$, so that $(\epsilon - 1)^\sharp \in p\mathbf{Z}_p^{\mathrm{cyc}}$. It follows that $\epsilon - 1$ is a pseudo-uniformizer of the tilt $(\mathbf{Q}_p^{\mathrm{cyc}})^\flat$: that is, we have

$$0 < |\epsilon - 1|_{(\mathbf{Q}_p^{\mathrm{cyc}})^\flat} < 1.$$

Let (K, ι) be an untilt of $C^\flat$ equipped with a (continuous) homomorphism $u : \mathbf{Q}_p^{\mathrm{cyc}} \hookrightarrow K$. Then the induced map $(\mathbf{Q}_p^{\mathrm{cyc}})^\flat \to K^\flat \simeq C^\flat$ carries ϵ to pseudo-uniformizer of the field $C^\flat$, which we will denote by $f(\epsilon)$.

Proposition 2. *The construction $(K, \iota, u) \mapsto u(\epsilon)$ induces a bijection*

$$\{\textit{Untilts } (K, \iota) \textit{ of } C^\flat \textit{ with an embedding } \mathbf{Q}_p^{\mathrm{cyc}} \hookrightarrow K\} \simeq \{x \in C^\flat : 0 < |x - 1|_{C^\flat} < 1\}.$$

Proof. Fix an element $x \in C^\flat$ satisfying $0 < |x - 1|_{C^\flat} < 1$. We wish to show that there is an essentially unique untilt K equipped with an embedding $u : \mathbf{Q}_p^{\mathrm{cyc}} \hookrightarrow K$ satisfying $(x^{1/p^n})^\sharp = f(\zeta_{p^n})$ for $n \geq 0$. Note that giving the embedding u is equivalent to choosing a compatible sequence of p^nth roots of unity in K, so we are just looking for an untilt K of $C^\flat$ having the property that $(x^{1/p})^\sharp$ is a primitive pth root of unity in K. This is equivalent to the requirement that $(x^{1/p})^\sharp$ satisfies the pth cyclotomic polynomial

$$1 + t + \cdots + t^{p-1} = 0;$$

that is, that the associated map $\theta : \mathbf{A}_{\mathrm{inf}} \to \mathcal{O}_K$ annihilates the element $\xi = 1 + [x^{1/p}] + \cdots + [x^{(p-1)/p}] \in \mathbf{A}_{\mathrm{inf}}$. Consequently, to prove the existence and uniqueness of K, it will suffice to show that ξ is a *distinguished* element of $\mathbf{A}_{\mathrm{inf}}$ (see Lecture 3). Writing $\xi = \sum_{n \geq 0} [c_n] p^n$, we wish to show that $|c_0|_{C^\flat} < 1$ and $|c_1|_{C^\flat} = 1$. Note that, since $x \equiv 1 \pmod{\mathfrak{m}_C^\flat}$ the image of ξ in the ring of Witt vectors $W(k) = W(\mathcal{O}_C^\flat / \mathfrak{m}_C^\flat)$ is p. We therefore have $|c_0|_{C^\flat} < 1$ and $|c_1 - 1|_{C^\flat} < 1$ (which is even more than we need). □

Note that, if K is an untilt of $C^\flat$ for which there exists *one* embedding $u : \mathbf{Q}_p^{\text{cyc}} \hookrightarrow K$, then we can always find many others: namely, we can precompose f with the automorphism of $\mathbf{Q}_p^{\text{cyc}}$ given by $\zeta_{p^n} \mapsto \zeta_{p^n}^\alpha$ for any $\alpha \in \mathbf{Z}_p^\times$. Under the correspondence of Proposition 2, this corresponds to the action of $\mathbf{Z}_p^\times$ on $\{x \in C^\flat : 0 < |x-1|_{C^\flat} < 1\}$ by exponentiation. We therefore obtain the following:

Corollary 3. *There is a canonical bijection*

$$\{\textit{Untilts } K \textit{ containing } p^n \textit{th roots of unity for all } n\}/\textit{isomorphism}$$
$$\downarrow \sim$$
$$\{x \in C^\flat : 0 < |x-1|_{C^\flat} < 1\}/\mathbf{Z}_p^\times.$$

Moreover, replacing an untilt (K, ι) by $(K, \iota \circ \varphi_C)$ has the effect of replacing the corresponding element $x \in C^\flat$ by $x^{1/p}$. We therefore have:

Corollary 4. *There is a canonical bijection*

$$\{\textit{Untilts } K \textit{ containing } p^n \textit{th roots of unity for all } n\}/\phi_{C^\flat}^{\mathbf{Z}}$$
$$\downarrow \sim$$
$$\{x \in C^\flat : 0 < |x-1|_{C^\flat} < 1\}/\,\mathbf{Q}_p^\times\,.$$

The inverse bijection carries the equivalence class of an element $x \in C^\flat$ to the "vanishing locus" of $\log([x]) \in B$

The proof is based on the following:

Exercise 5. Let K be a field of characteristic zero which is complete with respect to non-archimedean absolute value having residue characteristic p, so that the logarithm $\log(y) \in K$ is defined for $y \in K$ satisfying $|y-1|_K < 1$. Show that the construction $y \mapsto \log(y)$ induces a bijection

$$\{y \in K : |y-1|_K < |p|_K^{1/(p-1)}\} \to \{z \in K : |z| < |p|_K^{1/(p-1)}$$

(hint: the inverse is given by the exponential map $z \mapsto \exp(z) = \sum \frac{z^n}{n!}$).

Remark 6. The constant $|p|_K^{1/(p-1)}$ appearing in Exercise 5 is the best possible. Note that if K contains a primitive pth root of unity ζ_p, then ζ_p satisfies $|\zeta_p - 1|_K = |p|_K^{1/(p-1)} < 1$, and we have

$$p \log(\zeta_p) = \log(\zeta_p^p) = \log(1) = 0,$$

so that $\log(\zeta_p) = 0 = \log(1)$. It follows that the logarithm map is not injective when restricted to the *closed* disk $\{y \in K : |y-1|_K \leq |p|_K^{1/(p-1)}\}$.

Proof of Corollary 4. Let x be an element of $C^\flat$ satisfying $0 < |x-1|_{C^\flat} < 1$. Then, for any untilt (K, ι) of $C^\flat$, the element $x^\sharp$ satisfies $|(x^{p^n})^\sharp - 1|_K < |p|_K^{1/(p-1)}$ for $n \gg 0$. Consequently, if $\log(x^\sharp) = 0$, then $\log((x^{p^n})^\sharp) = 0$ and therefore $(x^{p^n})^\sharp = 1$ by virtue of Exercise **??**. Choose n as small as possible, so that $(x^{p^{n-1}})^\sharp \neq 1$ (this is possible since we have assumed that $x \neq 1$). Composing ι with a suitable power of the Frobenius map $\varphi_{C^\flat}$, we can assume that $n = 0$: that is, we have $x^\sharp = 1$ but $(x^{1/p})^\sharp \neq 1$, so that (K, ι) is the untilt associated to the element x in the proof of Proposition 2. □

Remark 7. One can show that when $C^\flat$ is algebraically closed, then every untilt of $C^\flat$ is also algebraically closed. It follows in this case that *every* Frobenius orbit of characteristic zero untilts of $C^\flat$ can be realized as the vanishing locus of some element of $B^{\varphi=p}$ having the form $\log([x])$; moreover, this element of $B^{\varphi=p}$ is unique up to the action of $\mathbf{Q}_p$.

Our next goal is to show that the functions of the form $\log([x])$ have *simple* zeros: that is, they do not vanish with multiplicity at any point of K. To make this idea precise, we need some auxiliary constructions.

Construction 8. Let $y = (K, \iota)$ be a characteristic zero untilt of $C^\flat$, so that we have a canonical surjection $\theta : \mathbf{A}_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_K$ whose kernel is a principal ideal (ξ). We saw in Lecture 3 that the ring $\mathbf{A}_{\mathrm{inf}}$ is ξ-adically complete: that is, it is isomorphic to the inverse limit of the tower

$$\cdots \to \mathbf{A}_{\mathrm{inf}}/(\xi^4) \twoheadrightarrow \mathbf{A}_{\mathrm{inf}}/(\xi^3) \twoheadrightarrow \mathbf{A}_{\mathrm{inf}}/(\xi^2) \twoheadrightarrow \mathbf{A}_{\mathrm{inf}}/(\xi) \simeq \mathcal{O}_K .$$

We let $B^+_{\mathrm{dR}} = B^+_{\mathrm{dR}}(y)$ denote the inverse limit of the diagram

$$\cdots \to (\mathbf{A}_{\mathrm{inf}}/(\xi^4))[\frac{1}{p}] \twoheadrightarrow (\mathbf{A}_{\mathrm{inf}}/(\xi^3))[\frac{1}{p}] \twoheadrightarrow \mathbf{A}_{\mathrm{inf}}/(\xi^2)[\frac{1}{p}] \twoheadrightarrow (\mathbf{A}_{\mathrm{inf}}/(\xi))[\frac{1}{p}] \simeq \mathcal{O}_K[\frac{1}{p}] = K.$$

Remark 9. The ring B^+_{dR} does not depend on the choice of generator ξ for the ideal $\ker(\theta)$: in each of the expressions above, we can replace the ideal (ξ^n) by $\ker(\theta)^n$. However, it does depend on the choice of untilt $y = (K, \iota)$.

Remark 10. In the situation of Construction 8, we can replace each of the localizations $(\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}]$ by $(\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{[\pi]}]$. Having made this replacement, the construction is sensible even when $K \simeq C^\flat$ has characteristic p. In this case, each quotient $\mathbf{A}_{\mathrm{inf}}(\xi^n)[\frac{1}{[\pi]}]$ can be identified with $W_n(\mathcal{O}^\flat_C)[\frac{1}{[\pi]}] \simeq W_n(\mathcal{O}^\flat_C[\frac{1}{\pi}]) \simeq W_n(C^\flat)$. Consequently, the "characteristic p" analogue of the ring B^+_{dR} is just the ring of Witt vectors $W(C^\flat)$.

Proposition 11. *In the situation of Construction 8, B^+_{dR} is a complete discrete valuation ring, and the element ξ is a uniformizer. In other words:*

- (a) *The image of ξ in B^+_{dR} is not a zero-divisor.*
- (b) *The ring B^+_{dR} is ξ-adically complete.*
- (c) *The quotient $B^+_{\mathrm{dR}}/(\xi)$ is a field.*

Proof. We first prove (a). Let f be an element of B^+_{dR}, given by a system of elements $x_n \in (\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}]$. We saw in Lecture 3 that ξ is not a zero divisor in $\mathbf{A}_{\mathrm{inf}}$ and p is not a zero-divisor in $\mathbf{A}_{\mathrm{inf}}/(\xi)$, and is therefore not a zero-divisor in $\mathbf{A}_{\mathrm{inf}}/(\xi^n)$ for all n. Consequently, we can view the quotient $\mathbf{A}_{\mathrm{inf}}/(\xi^n)$ as a subring of the localization $(\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}]$. We can therefore choose some $k \gg 0$ (depending on n) such that $p^k x_n$ belongs to $\mathbf{A}_{\mathrm{inf}}/(\xi^n)$. If $\xi x = 0$, then each $p^k x_n$ is annihilated by ξ in $\mathbf{A}_{\mathrm{inf}}/(\xi^n)$, so we can write $p^k x_n = \xi^{n-1} y_n$ for some $y_n \in \mathbf{A}_{\mathrm{inf}}/(\xi^n)$. Reducing modulo (ξ^{n-1}), we conclude that $p^k x_{n-1} = 0$ in $\mathbf{A}_{\mathrm{inf}}/(\xi^{n-1})$, so that $x_{n-1} = 0$. Since n is arbitrary, it follows that $x = 0$.

Note that each of the projection maps $B^+_{\mathrm{dR}} \twoheadrightarrow (\mathbf{A}_{\mathrm{inf}}/(\xi^m))[\frac{1}{p}]$ annihilates (ξ^m) and therefore factors as a surjection $\rho : B^+_{\mathrm{dR}}/(\xi^m) \twoheadrightarrow (\mathbf{A}_{\mathrm{inf}}/(\xi^m))[\frac{1}{p}]$. We claim that ρ is an isomorphism: that is, if x is an element of B^+_{dR} whose image in $(\mathbf{A}_{\mathrm{inf}}/(\xi^m))[\frac{1}{p}]$ vanishes, then x is divisible by ξ^m. Write $x = \{x_n\}_{n \geq 0}$ as above. For each $n \geq m$, we can choose $k(n) \gg 0$ such that $p^{k(n)} x_n \in \mathbf{A}_{\mathrm{inf}}/(\xi^n)$. Then the image of $p^{k(n)} x_n$ in $\mathbf{A}_{\mathrm{inf}}/(\xi^m)$ vanishes, so we can write $p^{k(n)} x_n = \xi^m y_n$ for some $y_n \in \mathbf{A}_{\mathrm{inf}}/(\xi^{n-m})$. Then x is the product of ξ^m with the element of B^+_{dR} given by the sequence $\{\frac{y_n}{p^{k(n)}}\}_{n \geq m}$.

It follows from the preceding argument that B_{dR}^{+} can be identified with the limit $\varprojlim B_{\mathrm{dR}}^{+}/(\xi^m)$ and is therefore ξ-adically complete. Moreover, in the case $m = 1$ we obtain an isomorphism $B_{\mathrm{dR}}^{+}/(\xi) \simeq K$ which proves (c). □

Remark 12. Since B_{dR}^{+} is a complete discrete valuation ring whose residue field $B_{\mathrm{dR}}^{+}/(\xi) \simeq K$ has characteristic zero, it is abstractly isomorphic to the formal power series ring $K[[\xi]]$. Beware that there is no canonical isomorphism of B_{dR}^{+} with $K[[\xi]]$.

Definition 13. We let B_{dR} denote the fraction field of the discrete valuation ring B_{dR}^{+} (that is, the ring $B_{\mathrm{dR}}^{+}[\frac{1}{\xi}]$).

Heuristically, if we think of the collection Y of all characteristic zero untilts $y = (K, \iota)$ of $C^{\flat}$ as as an analogue of the punctured unit disk $D^{\times}$ then $B_{\mathrm{dR}}^{+}(y)$ is the analogue is the analogue of the *completed local ring* $\widehat{\mathcal{O}}_{D^{\times},y}$ at a point $y \in D^{\times}$ (which is isomorphic to the power series ring $\mathbf{C}[[t]]$, where t is any local coordinate of $D^{\times}$ at the point y (for example, we can take $t = z - y$).

Note that, for every characteristic zero untilt $y = (K, \iota)$ of $C^{\flat}$, we have a canonical map $\mathbf{A}_{\mathrm{inf}} \to B_{\mathrm{dR}}^{+}$. Moreover, the composite map $\mathbf{A}_{\mathrm{inf}} \to B_{\mathrm{dR}}^{+} \to B_{\mathrm{dR}}^{+}/(\xi) \simeq K$ carries p and $[\pi]$ to invertible elements $p, \pi^{\sharp} \in K$. Consequently, the images of p and $[\pi]$ in B_{dR}^{+} are invertible: that is, we have a map $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}] \to B_{\mathrm{dR}}^{+}$, which we will denote by $f \mapsto \widehat{f}_y$.

Proposition 14. *Let $0 < a \leq b < 1$ be real numbers satisfying $a \leq |p|_K \leq b$. Then the map $f \mapsto \widehat{f}_y$ admits a canonical extension to a ring homomorphism $B_{[a,b]} \to B_{\mathrm{dR}}^{+}$, which we will also denote by $f \mapsto \widehat{f}_y$.*

Note that Proposition 14 is consistent with our heuristics: if we think of $B_{[a,b]}$ as the ring of "holomorphic" functions on the untilts K satisfying $a \leq |p|_K \leq b$, then we should be able to evaluate elements of $B_{[a,b]}$ not only *at* the point K (to obtain an element of K itself), but also "infinitesimally close" to K (to obtain an element of B_{dR}^{+}).

Proof. Without loss of generality, we may assume that $a = |p|_K = b$. Moreover, we can assume that the pseudo-uniformizer $\pi \in C^{\flat}$ is chosen so that $|\pi|_{C^{\flat}} = |p|_K$. In this case, the ring $B_{[a,b]}$ is obtained from the subring $\mathbf{A}_{\mathrm{inf}}[\frac{[\pi]}{p}, \frac{p}{[\pi]}]$ by first p-adically completing and then inverting the prime number p. For each $n \geq 0$, $\overline{e}$ determines a ring homomorphism

$$\overline{e}_n : \mathbf{A}_{\mathrm{inf}}[\frac{[\pi]}{p}, \frac{p}{[\pi]}] \to B_{\mathrm{dR}}^{+}/(\xi^n) \simeq (\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}].$$

We claim that there exists an integer $k \gg 0$ (depending on n) such that the image of $\overline{e}_n$ is contained in

$$p^{-k}(\mathbf{A}_{\mathrm{inf}}/(\xi^n)) \subseteq (\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}].$$

Assuming this is true, we can use the fact that $p^{-k}(\mathbf{A}_{\mathrm{inf}}/(\xi^n))$ is p-adically complete to extend $\overline{e}_n$ to a map (of abelian groups) from the p-adic completion of $\mathbf{A}_{\mathrm{inf}}[\frac{[\pi]}{p}, \frac{p}{[\pi]}]$ to $p^{-k}(\mathbf{A}_{\mathrm{inf}}/(\xi^n))$. Inverting p, we then obtain a map (of commutative rings)

$$e_n : B_{[a,b]} \to B_{\mathrm{dR}}^{+}/(\xi^n) \simeq (\mathbf{A}_{\mathrm{inf}}/(\xi^n))[\frac{1}{p}].$$

These maps are compatible as n varies, and determine the desired homomorphism $B_{[a,b]} \to B_{\mathrm{dR}}^{+}$.

It remains to prove the existence of k. Define $f, g \in B_{\mathrm{dR}}^{+}/(\xi^n)$ by the formulae

$$f = \overline{e}_n(\frac{[\pi]}{p}) \qquad g = f^{-1} = \overline{e}_n(\frac{p}{[\pi]}).$$

Our assumption that $|\pi|_{C^\flat} = |p|_K$ guarantees that the images of f and g under the map $B^+_{\mathrm{dR}}/(\xi^n) \to B^+_{\mathrm{dR}}/(\xi) \simeq K$ belong to the valuation ring $\mathcal{O}_K$. Consequently, we can find elements $f', g' \in \mathbf{A}_{\mathrm{inf}}/(\xi^n)$ satisfying

$$f \equiv f' \mod \xi \qquad g \equiv g' \mod \xi.$$

We therefore have

$$f = f' + \frac{\xi}{p^c} f'' \qquad g = g' + \frac{\xi}{p^c} g''$$

for some other elements $f'', g'' \in \mathbf{A}_{\mathrm{inf}}/(\xi^n)$ and some integer $c \gg 0$. In this case, every power of f admits a binomial expansion

$$\begin{aligned} f^m &= (f' + \frac{\xi}{p^c} f'')^m \\ &= \sum_{i=0}^{m} \binom{m}{i} f'^{m-i} (\frac{\xi}{p^c} f'')^i \\ &= \sum_{i=0}^{n-1} \binom{m}{i} f'^{m-i} (\frac{\xi}{p^c} f'')^i \\ &\in p^{-nc}(\mathbf{A}_{\mathrm{inf}}/(\xi^n)), \end{aligned}$$

and similarly with g in place of f. □