Lecture 7: The Artin-Hasse Exponential

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Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Fix an element $\pi \in C^\flat$ with $0 < |\pi|_{C^\flat} < 1$, and let B denote the completion of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the family of Gauss norms $|\bullet|_\rho$ for $0 < \rho < 1$. In the previous lecture, we showed that for each element $a \in \mathfrak{m}_C^\flat$, the infinite sum

$$\sum_{n \in \mathbf{Z}} \frac{[a^{p^n}]}{p^n}$$

converges to an element $x \in B$ satisfying $\varphi(x) = px$: that is, it is an element of the Frobenius eigenspace $B^{\varphi=p}$. We can now ask the following questions:

(1) Does every element of the eigenspace $B^{\varphi=p}$ have the form $\sum_{n \in \mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ for some element $a \in \mathfrak{m}_C^\flat$? If so, is the element a uniquely determined?

(2) Note that $B^{\varphi=p}$ is a vector space over $\mathbf{Q}_p$. Is the collection of elements of the form $\sum_{n \in \mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ closed under addition? If so, how do we add them?

Note that if every element of B were to admit a *unique* Teichmüller expansion, then the analysis of the previous lecture would give an affirmative answer to Question (1). We will eventually show that the answer to Question (1) is "yes" (even though we do not have existence and uniqueness results for Teichmüller expansions in general), but this will need to wait until we know a little bit more about the ring B. Our goal in this lecture is to show that, even without knowing the answer to (1), we can nevertheless answer Question (2) by describing the construction $a \mapsto \sum_{n \in \mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ in a different way.

Exercise 1. Let A be an algebra over $\mathbf{Q}_p$ equipped with a norm $|\bullet|_A$ satisfying the condition

$$|x \cdot y|_A \leq |x|_A \cdot |y|_A.$$

Let $x \in A$ be an element satisfying $|x - 1|_A < 1$. Show that the infinite sum

$$\log(x) = \sum_{k > 0} \frac{(-1)^{k+1}}{k}(x-1)^k$$

is a well-defined element of the completion $\widehat{A}$ (that is, the individual terms $\frac{(-1)^{k+1}}{k}(x-1)^k$ converge to zero as $k \to \infty$).

Assume that A is commutative, and let $y \in A$ be another element satisfying $|y - 1|_A < 1$. Show that xy satisfies $|xy - 1|_A < 1$ and $\log(xy) = \log(x) + \log(y)$ (in the completion $\widehat{A}$).

Example 2. Let x be an element of $C^\flat$ satisfying $|x - 1|_{C^\flat} < 1$. Note that $[x] - 1$ is an element of the ring $\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$, and therefore admits a Teichmüller expansion

$$[x] - 1 = \sum_{n \geq 0} [c_n] p^n$$

where the coefficients c_n satisfy $|c_n|_{C^\flat} \leq 1$. Moreover, we have $c_0 = x - 1$, so that $|c_0|_{C^\flat} < 1$. For each real number $\rho \in (0, 1)$, we have

$$|[x] - 1|_\rho = \sup\{|c_n|_{C^\flat} \rho^n\} \leq \max(|c_0|_{C^\flat}, \rho) < 1.$$

Applying Exercise 1, we conclude that the series

$$\log([x]) = \sum_{k>0} \frac{(-1)^{k+1}}{k}([x] - 1)^k$$

converges with respect to the Gauss norm $|\bullet|_\rho$. Since ρ is arbitrary, it follows that $\log([x])$ is a well-defined element of the ring B. Moreover, if y is another element of $C^\flat$ satisfying $|y - 1|_{C^\flat} < 1$, we have an identity

$$\log([xy]) = \log([x][y]) = \log([x]) + \log([y]).$$

Remark 3. For each $x \in 1 + \mathfrak{m}_C^\flat$, we have

$$\varphi(\log([x])) = \log(\varphi([x])) = \log([x^p]) = p\log([x]).$$

That is, $\log([x])$ actually belongs to the eigenspace $B^{\varphi=p} \subseteq B$.

We now have two explicit procedures for producing elements of the vector space $B^{\varphi=p}$: by forming Teichmüller expansions $\sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ (which converge for elements $a \in \mathfrak{m}_C^\flat$ which are "close to zero"), and by forming logarithms $\log([x])$ (which converge for elements $x \in 1 + \mathfrak{m}_C^\flat$ which are "close to one"). We will show that these procedures produce the same elements.

Theorem 4. *There exists a commutative diagram of sets*

$$\begin{tikzcd} & 1 + \mathfrak{m}_C^\flat \arrow[dr, "\log[\bullet]"] & \\ \mathfrak{m}_C^\flat \arrow[ur, "E"] \arrow[rr, "a \mapsto \sum \frac{[a^{p^n}]}{p^n}"] & & B^{\varphi=p}, \end{tikzcd}$$

where E is bijective.

Corollary 5. *The collection of elements of B of the form $\sum_{n\in\mathbf{Z}} \sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}$ is closed under addition.*

Let's assume for a moment that Theorem 4 is true, and try to guess the nature of the function E. It follows from Theorem 4 that we can define an addition law $\bigoplus$ on the set $\mathfrak{m}_C^\flat$, having the property that

$$\sum_{n\in\mathbf{Z}} \frac{[(a \bigoplus b)^{p^n}]}{p^n} = (\sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n}) + (\sum_{n\in\mathbf{Z}} \frac{[b^{p^n}]}{p^n}).$$

Namely, we define $a \oplus b = E^{-1}(E(a) \cdot E(b))$, so that we have an identity $E(a \oplus b) = E(a) \cdot E(b)$. We can therefore think of E as something like an exponential map, which relates the modified addition law $\bigoplus$ on $\mathfrak{m}_C^\flat$ (related to the addition of Teichmüller expansions) to the usual multiplication on $1 + \mathfrak{m}_C^\flat$.

Lemma 6. *Let $\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$ be the power series for the exponential function, regarded as an element of $\mathbf{Q}[[x]]$. Then we have an identity of formal power series*

$$\exp(x) = \prod_{d>0} (\frac{1}{1 - x^d})^{\frac{\mu(d)}{d}}.$$

Here μ denotes the Möbius function

$$\mu(d) = \begin{cases} (-1)^n & \text{if } d = p_1 \cdots p_n \text{ for distinct primes } p_1, p_2, \ldots, p_n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Taking the logarithm of the right hand side yields

$$\begin{aligned}
\log(\prod_{d>0}(\frac{1}{1-x^d})^{\frac{\mu(d)}{d}} &= \sum_{d>0}\log((\frac{1}{1-x^d})^{\frac{\mu(d)}{d}}) \\
&= \sum_{d>0}\frac{\mu(d)}{d}\log(\frac{1}{1-x^d}) \\
&= \sum_{d>0}\frac{\mu(d)}{d}\sum_{d'>0}\frac{x^{d'd}}{d'} \\
&= \sum_{n>0}\frac{x^n}{n}\sum_{d|n}\mu(d) \\
&= x.
\end{aligned}$$

where the final equality follows from the identity

$$\sum_{d|n}\mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

□

The exponential function $x \mapsto \exp(x)$ has good convergence properties over the real numbers (where the coefficients $\frac{1}{n!}$ are small), but much weaker convergence properties when working p-adically (where the coefficients $\frac{1}{n!}$ are large). However, we can get a function with better p-adic behavior by "leaving out" the problematic terms in the product decomposition of Lemma 6.

Definition 7. Fix a prime number p. The *Artin-Hasse exponential* $E(x)$ is the power series

$$E(x) = \prod_{(d,p)=1}(\frac{1}{1-x^d})^{\frac{\mu(d)}{d}},$$

where the product is taken over the collection of all positive integers d which are relatively prime to p.

Note that the coefficients of this power series are integral at p: that is, we can think of $E(x)$ as a power series with coefficients in the subring $\mathbf{Z}_{(p)} \subseteq \mathbf{Q}$, given by (in contrast with the usual exponential series $\exp(x)$).

Exercise 8. Show that, as a formal power series with rational coefficients, the Artin-Hasse exponential $E(x)$ is given by the formula

$$E(x) = \exp(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots).$$

(Hint: take the logarithm of both sides and argue as in Lemma 6).

Since the power series $E(x) = 1 + x + \text{ higher order terms }$ has coefficients in $\mathbf{Z}_{(p)}$, the construction $a \mapsto E(a)$ determines a bijection from $\mathfrak{m}_C^\flat$ to $1+\mathfrak{m}_C^\flat$. We claim that this bijection satisfies the requirements of Theorem 4. In other words, we claim that for each element $a \in \mathfrak{m}_C^\flat$, we have an equality

$$\sum_{n\in\mathbf{Z}}\frac{[a^{p^n}]}{p^n} = \log([E(a)]) = \log([\prod_{(d,p)=1}(\frac{1}{1-a^d})^{\frac{\mu(d)}{d}}])$$

in the ring B. We will establish this identity by manipulation of formal series, and leave it to the reader to justify that our manipulations are legal (that is, that all of the infinite sums and products that we consider are convergent with respect to each of the Gauss norms).

We first recall that for any element $y \in \mathcal{O}_C^\flat$, the Teichmüller representative $[y] \in W(\mathcal{O}_C^\flat)$ can be computed as the limit $\varinjlim_{k\to\infty} \widetilde{y}_k^{p^k}$, where $\widetilde{y}_k$ is any element of $W(\mathcal{O}_C^\flat)$ lying over $y^{p^{-k}}$. In particular, for $x \in \mathfrak{m}$, we have

$$[1-x] = \lim_{k\to\infty} (1 - [x^{p^{-k}}])^{p^k}$$

$$\begin{aligned} \log \frac{1}{[1-x]} &= \lim_{k\to\infty} p^k \log(\frac{1}{1-[x^{p^{-k}}]}) \\ &= \lim_{k\to\infty} p^k \sum_{m>0} \frac{[x^{mp^{-k}}]}{m} \\ &= \sum_{\alpha\in\mathbf{Z}[1/p], \alpha>0} \frac{[x^\alpha]}{\alpha}. \end{aligned}$$

We now write

$$\begin{aligned} \log[\prod_{(d,p)=1} (1-a^d)^{\frac{-\mu(d)}{d}}] &= \sum_{(d,p)=1} \frac{\mu(d)}{d} \log \frac{1}{[1-a^d]} \\ &= \sum_{(d,p)=1} \sum_{\alpha\in\mathbf{Z}[1/p],\alpha>0} \mu(d) \frac{[a^{d\alpha}]}{d\alpha} \\ &= \sum_{\beta\in\mathbf{Z}[1/p],\beta>0} \sum_d \mu(d) \frac{[a^\beta]}{\beta}, \end{aligned}$$

where, in the final expression, we write $\beta = p^n k$ for $(k,p) = 1$ and d ranges over all divisors of k. It follows from Equation (1) that this inner sum vanishes for $k \neq 1$: that is, we can neglect all values of β which are not powers of p. Doing so, we obtain the expression

$$\sum_{n\in\mathbf{Z}} \frac{[a^{p^n}]}{p^n},$$

as desired.