Lecture 5: Norms

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Our goal in this lecture is to describe another way of thinking about some of the rings appearing in the previous lecture. First, we review some topological algebra.

Definition 1. Let V be a topological vector space over $\mathbf{Q}_p$. We say that V is a *p-adic Banach space* if there exists an open $\mathbf{Z}_p$-submodule $V_0 \subseteq V$, which is closed under addition, such that V_0 is p-adically complete as an abelian group and satisfies $V = V_0[\frac{1}{p}]$.

Example 2. Fix a non-archimedean norm $| \bullet |_{\mathbf{Q}_p}$ on $\mathbf{Q}_p$, compatible with the usual topology. For example, we can take the usual p-adic norm, characterized by $|p|_{\mathbf{Q}_p} = \frac{1}{p}$; however, it will be convenient not to assume this.

Let V be a vector space over $\mathbf{Q}_p$. We define a *norm* on V to be a function $| \bullet |_V : V \to \mathbf{R}_{\geq 0}$ satisfying

$$|\lambda v|_V = |\lambda|_{\mathbf{Q}_p} \cdot |v|_V \qquad |v + w|_V \leq \max(|v|_V, |w|_V)$$

(this is sometimes called a *pre-norm*, with the term *norm* reserved for the case where $|v|_V = 0 \Rightarrow v = 0$).

Any norm on V equips V with the structure of a (pre)metric space, with metric $d(v, w) = |v - w|$. If V is separated and complete with respect to this metric, then it is a p-adic Banach space (take $V_0 = \{v \in V : |v|_V \leq 1\}$ to be the "unit ball" of V).

Remark 3. Every p-adic Banach space V can be obtained from the construction of Example 2. Let $V_0 \subseteq V$ be an open $\mathbf{Z}_p$-module which is p-adically complete. We can then define a map $| \bullet |_V : V \to \mathbf{R}_{\geq 0}$ by the formula

$$|v|_V = \inf\{|\lambda|_{\mathbf{Q}_p} : v \in \lambda \cdot V_0\};$$

this is a norm on V, having V_0 as the unit ball.

Example 4. Let V be a vector space over $\mathbf{Q}_p$ equipped with a pair of norms $| \bullet |_V$ and $| \bullet |'_V$ (possibly with respect to different choices of absolute value $| \bullet |_{\mathbf{Q}_p}$ and $| \bullet |'_{\mathbf{Q}_p}$. We can then regard V as a metric space with respect to the metric $d(v, w) = |v - w|_V + |v - w|'_V$. If V is complete with respect to this metric, then it is a p-adic Banach space (the intersection of unit balls $V_0 = \{v \in V : |v|_V \leq 1 \text{ and } |v|'_V \leq 1\}$ satisfies the requirements of Definition 1).

Alternatively, in the case $| \bullet |_{\mathbf{Q}_p} = | \bullet |'_{\mathbf{Q}_p}$ (which we can always arrange by raising to an appropriate power), we can equip V with the norm $v \mapsto |v|_V + |v|_{V'}$.

Example 5. Let K be any completely valued field of characteristic zero and residue characteristic p. Then K is a p-adic Banach space.

Example 6. Let V_0 be any abelian group which is p-adically complete and p-torsion free. Then V_0 has the structure of a module over the ring $\mathbf{Z}_p$, and the tensor product $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} V_0 = V_0[\frac{1}{p}]$ can be regarded as a p-adic Banach space (by equipping it with the topology where the subsets $p^n V_0$ form a neighborhood basis of the identity).

Example 7. Let M be an abelian group which is p-torsion free. We can then apply the construction of Example 6 to the p-adic completion $\widehat{M} = \varprojlim M/p^n M$ to obtain a p-adic Banach space $\widehat{M}[\frac{1}{p}]$.

Example 8. Let V be a $\mathbf{Q}_p$-vector space equipped with a norm. Then the completion of V (as a metric space) is a $\mathbf{Q}_p$-Banach space.

Examples 7 and 8 are related. If V is a $\mathbf{Q}_p$-vector space equipped with a norm, then the unit ball $V_0 = \{v \in V : |v|_V \leq 1\}$ is a p-torsion free abelian group. The completion of V with respect to its norm can then be identified with $\widehat{V}_0[\frac{1}{p}]$, where $\widehat{V}_0$ is the p-adic completion of V_0.

Variant 9. *Suppose that V is equipped with a pair of norms $|\bullet|_V$ and $|\bullet|_{V'}$. Then the completion of V with respect to the metric $d(v,w) = |v-w|_V + |v-w|'_V$ is given by $\widehat{V}_0[\frac{1}{p}]$, where $V_0 = \{v \in V : |v|_V \leq 1, |v|'_V \leq 1\}$.*

Let us now turn to the example of interest to us. Fix a perfectoid field $C^\flat$, with valuation ring $\mathcal{O}_C^\flat$ and set $\mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat)$. Fix an element $\pi \in C^\flat$ satisfying $0 < |\pi|_{C^\flat} < 1$ and consider the localization $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. Every element of this ring admits a Teichmüller expansion

$$\sum_{n \gg -\infty} [c_n] p^n$$

where the coefficients $c_n \in C^\flat$ are bounded.

Definition 10. [Gauss Norms] Fix a real number $0 < \rho < 1$. For each element $f = \sum_{n \gg -\infty} [c_n] p^n \in \mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$, we define

$$|f|_\rho = \sup\{|c_n|_{C^\flat} \cdot \rho^n\}.$$

Remark 11. In the situation of Definition 10, the real numbers $|c_n|_{C^\flat} \cdot \rho^n$ vanish for $n \ll 0$ and decay exponentially as $\rho \to \infty$. Consequently, the supremum $\sup\{|c_n|_{C^\flat} \cdot \rho^n\}$ is exists and is realized by finitely many values of n.

Notation 12. We let Y denote the set of all isomorphism classes of characteristic zero untilts (K, ι) of $C^\flat$. We will use the letter y to denote a typical point of Y, given by an untilt (K, ι) of $C^\flat$. For every such point y, we have a surjective ring homomorphism

$$\theta_y : \mathbf{A}_{\text{inf}} \to \mathcal{O}_K \qquad \sum_{n \geq 0} [c_n] p^n \mapsto \sum_{n \geq 0} c_n^\sharp p^n$$

which extends to a ring homomorphism $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}] \to K$. We denote the value of this homomorphism on an element $f \in \mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ by $f(y) \in K$.

Given $0 < a \leq b < 1$, we let $Y_{[a,b]} \subseteq Y$ denote the subset consisting of those points $y = (K, \iota)$ satisfying $a \leq |p|_K \leq b$.

Remark 13. Let $y = (K, \iota)$ be a point of Y and let $\rho = |p|_K$. Then, for every element $f = \sum_{n \gg -\infty} [c_n] p^n \in \mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$, we have

$$|f(y)|_K = |\sum_{n \gg -\infty} c_n^\sharp p^n|_K \leq \sup\{|c_n^\sharp|_K \cdot |p|_K^n\} = \sup\{|c_n|_{C^\flat} \cdot \rho^n\} = |f|_\rho.$$

Remark 14. Let $f = \sum_{n \gg -\infty} [c_n] p^n$ be an element of $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}$. We will say that a real number $\rho \in (0,1)$ is *generic for* f if the supremum $\sup\{|c_n|_{C^\flat} \rho^n\}$ is achieved exactly once. That is, ρ is generic for f if there is an integer n such that $|f|_\rho = |c_n|_{C^\flat} \rho^n$, and for all integers $m \neq n$ we have $|c_m|_{C^\flat} \rho^m < |f|_\rho$. In this case, if $y = (K, \iota)$ is a point of Y satisfying $|p|_K = \rho$, the inequality of Remark 13 can be replaced by an equality $|f|_\rho = |f(y)|_K$.

Exercise 15. Let f be an element of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. Show that the set

$$\{\rho \in (0,1) : \ \rho \text{ is not generic for } f \ \}$$

is a discrete subset of $(0,1)$. In other words, if ρ is not generic for f, then $\rho \pm \epsilon$ *will* be generic for f for all sufficiently small $\epsilon \neq 0$.

Proposition 16. *For each $0 < \rho < 1$, the map $|\bullet|_\rho : \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}] \to \mathbf{R}_{\geq 0}$ is a norm (in the sense of Example 2), compatible with the norm on $\mathbf{Q}_p$ satisfying $|p|_{\mathbf{Q}_p} = \rho$.*

Proof. We first show that $|f+g|_\rho \leq \max(|f|_\rho, |g|_\rho)$. Write $f + g = \sum_{n \gg -\infty} [c_n] p^n$. Suppose that the following conditions are satisfied:

($*$) The real number ρ is generic for f and belongs to the value group of $C^\flat$.

In this case, we can choose a point $y = (K, \iota) \in Y$ satisfying $|p|_K = \rho$ (for example, by taking $\mathcal{O}_K = \mathbf{A}_{\inf}/([c] - p)$, where $c \in C^\flat$ is any element satisfying $|c|_{C^\flat} = \rho$). Remark 14 then gives

$$|f+g|_\rho = |(f+g)(y)|_K \leq \max(|f(y))|_K, |g(y)|_K) \leq \max(|f|_\rho, |g|_\rho).$$

It follows from Exercise 15 that the collection of real numbers ρ satisfying ($*$) is dense in $(0,1)$. Consequently, it follows by continuity that $|f+g|_\rho \leq \max(|f|_\rho, |g|_\rho)$ for all $\rho \in (0,1)$.

It follows for that each $f \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ and each integer n, we have $|nf|_\rho \leq |f|_\rho$. By a continuity argument, we conclude that $|\lambda f|_\rho \leq |f|_\rho$ for each $\lambda \in \mathbf{Z}_p$. If λ is an invertible element of $\mathbf{Z}_p$, then the same argument gives $|f|_\rho \leq |\lambda f|_\rho$, so that $|\lambda f|_\rho = |f|_\rho = |\lambda|_{\mathbf{Q}_p} \cdot |f|_\rho$. Since every nonzero element of $\mathbf{Q}_p$ factors as $p^n u$, where u is an invertible element of $\mathbf{Z}_p$, we are reduced to checking the identity $|\lambda f|_\rho = |\lambda|_{\mathbf{Q}_p} \cdot |f|_\rho$ in the case $\lambda = p$: that is, the identity $|pf|_\rho = \rho \cdot |f|_\rho$. This follows immediately from the definition. □

Variant 17. *For every pair of elements $f, g \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$, we have $|f \cdot g|_\rho = |f|_\rho \cdot |g|_\rho$.*

Proof. Assume first that the following condition is satisfied:

($*$) The element ρ is generic for f, g, and $f \cdot g$, and belongs to the value group of $C^\flat$.

As in the proof of Proposition 16, we can choose a point $y = (K, \iota) \in Y$ satisfying $|p|_K = \rho$. In this case, Remark 13 gives

$$|f \cdot g|_\rho = |(f \cdot g)(y)|_K = |f(y)|_K \cdot |g(y)|_K = |f|_\rho \cdot |g|_\rho.$$

We conclude by observing that the collection of real numbers $\rho \in (0,1)$ satisfying ($*$) is dense, so by continuity we have an equality $|fg|_\rho = |f|_\rho \cdot |g|_\rho$ for all $\rho \in (0,1)$. □

Proposition 18. *Suppose that a and b belong to the value group of $C^\flat$, so that we can choose elements $\pi_a, \pi_b \in C^\flat$ satisfying $|\pi_a|_{C^\flat} = a$ and $|\pi_b|_{C^\flat} = b$. Then the intersection of unit balls*

$$V_0 = \{f \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]} : |f|_a \leq 1, |f|_b \leq 1\}$$

is the subring $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ of the previous lecture.

Proof. It follows from Proposition 16 and Variant 17 that V_0 is a subring of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. This subring clearly contains $\mathbf{A}_{\inf}$: note that if $f = \sum_{n \geq 0} [c_n] p^n$ belongs to $\mathbf{A}_{\inf}$, then we automatically have

$$|f|_\rho = \sup_{n \geq 0} \{|c_n|_{C^\flat} \cdot \rho^n\} \leq 1$$

for any $0 < \rho < 1$. Moreover, it also contains $\frac{[\pi_a]}{p}$ and $\frac{p}{[\pi_b]}$, by virtue of the equalities

$$|\frac{[\pi_a]}{p}|_a = 1 \qquad |\frac{[\pi_a]}{p}|_b = \frac{a}{b} < 1$$

$$|\frac{p}{[\pi_b]}|_a = \frac{a}{b} < 1 \qquad |\frac{p}{[\pi_b]}|_b = 1.$$

This shows that $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ is contained in V_0.

We now prove the reverse containment. Suppose that $f = \sum_{n \gg -\infty} [c_n] p^n$ satisfies $|f|_a \leq 1$ and $|f|_b \leq 1$; we wish to show that f belongs to $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$. By assumption, the absolute values $|c_n|_{C^\flat}$ are bounded above. We may therefore choose some integer $m \gg 0$ such that each product $\pi_b^m c_n$ belongs to $C^\flat$. We then have

$$f = (\sum_{n<m} [c_n] p^n) + (\sum_{n \geq 0} [c_{n+m} \pi_b^m] p^n) (\frac{p}{[\pi_b]})^m$$

where the second summand belongs to $\mathbf{A}_{\inf}[\frac{p}{[\pi_b]}]$ (and therefore also to the unit ball of $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$). Subtracting, we can reduce to the case where the Teichm"uller expansion of f is finite.

Our assumption that $|f|_a \leq 1$ and $|f|_b \leq 1$ guarantees that, for each integer n, we have

$$|c_n|_{C^\flat} \cdot a^n \leq 1 \qquad |c_n|_{C^\flat} \cdot b^n \leq 1,$$

so that $c_n \pi_a^n$ and $c_n \pi_b^n$ belong to $\mathcal{O}_C^\flat$. For $n \leq 0$, this implies that $[c_n] p^n = [c^n \pi_a^n] (\frac{[\pi_a]}{p})^{-n}$ belongs to $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}]$. For $n \geq 0$, we instead learn that $[c_n] p^n = [c_n \pi_b^n] (\frac{p}{[\pi_b]})^n$ belongs to $\mathbf{A}_{\inf}[\frac{p}{[\pi_b]}]$. It follows that f belongs to the ring $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$, as desired. □

Corollary 19. *Suppose that a and b belong to the value group of $C^\flat$. Then the ring $B_{[a,b]}$ of the previous lecture can be identified with the completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to the pair of norms $| \bullet |_a$ and $| \bullet |_b$.*

We will henceforth use this Corollary to extend the definition of $B_{[a,b]}$ to the case where a and b do not necessarily belong to the value group of $C^\flat$). Note that if $y = (K, \iota) \in Y$ is an untilt satisfying $a \leq |p|_K \leq b$, then Remark 13 implies that the homomorphism

$$\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}] \to K \qquad f \mapsto f(y)$$

admits a continuous extension $B_{[a,b]} \to K$, which we will also denote by $f \mapsto f(y)$.