Lecture 4: Holomorphic Functions of the Variable p

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Throughout this lecture, we let $C^\flat$ denote a perfectoid field of characteristic p and $\mathcal{O}_C^\flat$ its valuation ring. As in the previous lecture, we are interested in classifying *untilts* of $C^\flat$: that is, perfectoid fields K equipped with an isomorphism $\iota : C^\flat \simeq K^\flat$

Exercise 1. Let K be a field. Suppose we are given two non-archimedean absolute values $|\bullet|_K, |\bullet|'_K : K \to \mathbf{R}_{\geq 0}$. Show that the following conditions are equivalent:

(a) The absolute values $|\bullet|_K$ and $|\bullet|'_K$ determine the same topology on K (one given by the metric $d(x,y) = |x-y|_K$, the other by $d'(x,y) = |x-y|'_K$.

(b) The valuation rings $\mathcal{O}_K = \{x \in K : |x|_K \leq 1\}$ and $\mathcal{O}'_K = \{x \in K : |x|'_K \leq 1\}$ are the same.

(c) There exists a constant $\alpha > 0$ such that $|x|'_K = |x|^\alpha_K$ for all $x \in K$ (moreover, if the valuations $|\bullet|_K$ and $|\bullet|'_K$ are nontrivial, then α is unique).

In general, when speaking of a valued field K, we will assume only that the topology of K is given; the absolute value $|\bullet|_K$ itself is only well-defined up to a constant exponent. If K is of characteristic zero with residue characteristic p, then there is a canonical way to normalize the absolute value of K: we can demand that, on the subfield $\mathbf{Q} \subseteq K$, the absolute value $|\bullet|_K$ restricts to the usual p-adic absolute value (so that $|p|_K = \frac{1}{p}$). On the other hand, if K is given to us as an untilt of $C^\flat$ (and we have fixed a normalization for the absolute value on $C^\flat$), then there is another way to normalize the absolute value of K: we can equip K with the unique absolute value for which the isomorphism ι satisfies $|x|_{C^\flat} = |\iota(x)|_{K^\flat} = |\iota(x)^\sharp|_K$. In general, these normalizations are different. We can exploit this to construct an invariant:

Construction 2. Let (K, ι) be an untilt of $C^\flat$. We let $r(K,\iota)$ denote the real number given by $|p|_K$, where $|\bullet|_K$ has been normalized so that ι is compatible with absolute values.

Remark 3. The real number r depends on a choice of absolute value on $C^\flat$; modifying the absolute value by some exponent α has the effect of replacing r by r^α.

Note that we have $0 \leq r(K,\iota) < 1$. Moreover, up to isomorphism, there is only one untilt (K,ι) satisfying $r(K,\iota) = 0$: namely, the characteristic p untilt given by $K \simeq C^\flat$.

Heuristic Idea 4. The collection of all (isomorphism classes of) untilts of $C^\flat$ behaves in some respects like the unit disk $\{z \in \mathbf{C} : |z| < 1\}$ in the complex plane. Here the function $(K,\iota) \mapsto r(K,\iota) = |p|_K$ is analogous to the function $z \mapsto |z|$, and the characteristic p untilt $(C^\flat, \mathrm{id})$ is analogous to the complex number 0.

This heuristic is supported by some of the constructions of the previous lecture. Recall that $\mathbf{A}_{\mathrm{inf}}$ denotes the ring of Witt vectors $W(\mathcal{O}_C^\flat)$. Since $\mathcal{O}_C^\flat$ is a perfect ring of characteristic p, every element $x \in \mathbf{A}_{\mathrm{inf}}$ admits a unique Teichmüller expansion

$$x = [c_0] + [c_1]p + [c_2]p^2 + [c_3]p^3 + \cdots$$

It is useful to think of the elements of $\mathbf{A}_{\mathrm{inf}}$ can be viewed as "power series in the variable p" which can be evaluated in any untilt K of $C^\flat$ by applying the homomorphism $\theta : \mathbf{A}_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_K$ of the previous lecture, thereby producing an element

$$\theta(x) = c_0^\sharp + c_1^\sharp p + c_2^\sharp p^2 + \cdots \in \mathcal{O}_K,$$

just as a power series $f(z) = \sum_{n \geq 0} c_n z^n$ with complex coefficients satisfying $|c_n| \leq 1$ can be evaluated at any complex number z satisfying $|z| < 1$.

It will be useful to consider various enlargements of the ring $\mathbf{A}_{\mathrm{inf}}$. Fix a quasi-uniformizer $\pi \in C^\flat$: that is, an element of $C^\flat$ satisfying $0 < |\pi|_{C^\flat} < 1$. For each untilt K of $C^\flat$, the image $\pi^\sharp$ is a quasi-uniformizer in K: that is, the map

$$\mathbf{A}_{\mathrm{inf}} \to \mathcal{O}_K$$

extends to a map of localizations

$$\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\pi]}] \to \mathcal{O}_K[\frac{1}{\pi^\sharp}] = K.$$

Note that the localization on the left hand side is independent of π. Concretely, each element of the localization $\mathbf{A}_{\mathrm{inf}}$ admits a unique Teichmüller expansion

$$\sum_{n \geq 0} [c_n] p^n,$$

where each c_n belongs to $C^\flat$ and the sequence $\{|c_n|\}_{n \geq 0}$ is bounded; here $\sum_{n \geq 0} [c_n] p^n$ is defined as the fraction

$$\frac{\sum_{n \geq 0} [c_n \pi^m] p^n}{[\pi^m]}$$

for m sufficiently large.

If we are interested in functions that we can evaluate only on *characteristic zero* untilts of $C^\flat$, there is a further enlargement we can make: for each characteristic zero untilt K of $C^\flat$, the map

$$\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\pi]}] \to K$$

factors through the localization $\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\pi]}, \frac{1}{p}]$. Once again, elements of this localization have a concrete description: they admit Teichmüller expansions

$$\sum_{-\infty < n < \infty} [c_n] p^n,$$

where we allow negative powers of p but require that $c_n = 0$ for $n \ll 0$ (and that the set $\{|c_n|\}$ is bounded).

We can extend Heuristic 4 as follows:

Perfectoid Geometry	Complex Analysis
$\{\mathrm{Untilts}(K,\iota)\}/\sim$	Open disk $\{z \in \mathbf{C} : \lvert z\rvert < 1\}$
Real number $r(K,\iota) = \lvert p\rvert_K$	Real number $\lvert z\rvert$
Prime number p	Coordinate function z
Period ring $\mathbf{A}_{\mathrm{inf}}$	Power series $\sum_{n\geq 0} c_n z^n$, $\lvert c_n\rvert \leq 1$
$\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}]$	Laurent series $\sum_{n\geq -k} c_n z^n$, $\lvert c_n\rvert \leq 1$
$\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\pi]}]$	Power series $\sum_{n\geq 0} c_n z^n$, $\lvert c_n\rvert$ bounded
$\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$	Laurent series $\sum_{n\geq -k} c_n z^n$, $\lvert c_n\rvert$ bounded

The collections of power series that appear on the right hand side are a bit strange looking (in the setting of complex analysis). Note if $f(z) = \sum_{n\geq 0} c_n z^n$ is a power series with complex coefficients, then the condition that the coefficients c_n are bounded is sufficient, but not necessary, to guarantee that $f(z)$ converges in the entire unit disk $\{z \in \mathbf{C} : |z| < 1\}$. However, the collection of such power series is not closed under addition or multiplication. It is better to look instead at the collection of power series

$$\{\sum_{n\geq 0} c_n z^n : \limsup |c_n|^{1/n} \leq 1\},$$

which are exactly the Taylor expansions of holomorphic functions on $\{z \in \mathbf{C} : |z| < 1\}$. Also, the Laurent series $\sum_{n\geq -k} c_n z^n$ appearing on the right hand side are actually *meromorphic* at $z = 0$. This has a natural enlargement, where we consider the ring of *all* holomorphic functions on the punctured unit disk $\{z \in \mathbf{C} : 0 < |z| < 1\}$. Our goal over the next few lectures will be to study an analogue of this ring of holomorphic functions in perfectoid geometry, which we will denote by B. The ring B will be a certain enlargement of the ring $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$, whose elements can be viewed heuristically as "holomorphic functions of p." Before giving a definition of B, let us give an example of the sort of function that we would like it to contain.

Exercise 5. Let K be a completely valued field of characteristic zero. Show that, for every element $x \in \mathfrak{m}_K$, the power series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

converges.

Construction 6. Let e be an element of $\mathcal{O}_C^\flat$ satisfying $|e-1|_{C^\flat} < 1$. Then, for each untilt K of $C^\flat$, the image $e^\sharp$ satisfies $|e^\sharp - 1|_K < 1$. If K has characteristic zero, then it follows from Exercise 5 that the logarithm

$$\log(e^\sharp) = \log(1 + (e^\sharp - 1)) = \sum_{n>0} (-1)^{n+1} \frac{(e^\sharp - 1)^n}{n}$$

is a well-defined element of K.

It is tempting to denote the function $(K,\iota) \mapsto (\log(e^\sharp) \in K)$ by $\log([e])$. However, the expression $\log([e])$ has no meaning in the ring of Witt vectors $\mathbf{A}_{\mathrm{inf}}$, or even in the localization $\mathbf{A}_{\mathrm{inf}}[1/p, \frac{1}{[\pi]}]$; it has an "essential singularity" at $p = 0$.

The definition of the ring B is somewhat complicated. It will be defined as a certain inverse limit of rings $B_{[a,b]}$, where a and b are (certain) real numbers satisfying $0 < a \leq b < 1$. The idea is that $B_{[a,b]}$ should consist of "holomorphic functions of p" which are defined on the strip $a \leq |p|_K \leq b$.

Construction 7. Let a and b be real numbers satisfying $0 < a \leq b < 1$, and suppose that a and b belong to the *value group* of the field $C^\flat$: that is, there exist elements $\pi_a, \pi_b \in C^\flat$ satisfying $|\pi_a|_{C^\flat} = a$ and $|\pi_b|_{C^\flat} = b$ (these elements then belong to the maximal ideal $\mathfrak{m}_C^\flat$, and are well-defined up to multiplication by units in $\mathcal{O}_C^\flat$).

We let $\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ denote the subring of $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ generated by $\mathbf{A}_{\text{inf}}$ together with the elements $\frac{[\pi_a]}{p}$ and $\frac{p}{[\pi_b]}$. Note that this subring depends only on the real numbers a and b, and not on the elements π_a and π_b.

Let $\widehat{\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]}$ denote the p-adic completion of $\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ in the usual algebraic sense: that is, the ring

$$\varprojlim_n \mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]/p^n \mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}].$$

We define $B_{[a,b]} = \widehat{\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]}[\frac{1}{p}]$.

Remark 8. Note that since $\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ contains the element $x = \frac{p}{[\pi_b]}$ satisfying $x \cdot [\pi_b] = p$, the element $[\pi_b]$ becomes invertible in the ring $B_{[a,b]}$. Since π divides some power of π_b, the element $[\pi]$ also becomes invertible in $B_{[a,b]}$. Consequently, we can view $B_{[a,b]}$ as an algebra over $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. In fact, $B_{[a,b]}$ can be viewed as a completion of $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ with respect to a certain Banach norm; we will return to this viewpoint in the next lecture.

Remark 9. Let K be an untilt of $C^\flat$, and suppose that $a \leq |p|_K \leq b$ (so that, in particular, K has characteristic zero). It follows that $|\pi_a^\sharp|_K \leq |p|_K \leq |\pi_b^\sharp|_K$, so that $\frac{\pi_a^\sharp}{p}$ and $\frac{p}{\pi_b^\sharp}$ belong to $\mathcal{O}_K$. Consequently, the canonical map $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}] \to K$ carries the subring $\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ into $\mathcal{O}_K$. Since $\mathcal{O}_K$ is p-adically complete, this map extends over the completion $\widehat{\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]}$. Inverting p, we obtain a map

$$B_{[a,b]} = \widehat{\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]}[\frac{1}{p}] \to \mathcal{O}_K[\frac{1}{p}] = K.$$

These maps fit into a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{A}_{\text{inf}} & \hookrightarrow & \mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}] & \longrightarrow & \widehat{\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]} & \longrightarrow & \mathcal{O}_K \\
& \searrow & \downarrow & & \downarrow & & \downarrow \\
& & \mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}] & \longrightarrow & B_{[a,b]} & \longrightarrow & K.
\end{array}$$

Remark 10. Suppose we are given another pair of real numbers $0 < a' \leq b' < 1$ belonging to the value group of $C^\flat$, so that we can write $a' = |\pi_{a'}|_{C^\flat}$ and $b' = |\pi_{b'}|_{C^\flat}$. Suppose that the closed interval $[a', b']$ is contained in $[a, b]$: that is, we have $a \leq a' \leq b' \leq b$. Then π_a is divisible by $\pi_{a'}$, and $\pi_{b'}$ is divisible by π_b. It follows that the ring $\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$ is contained in $\mathbf{A}_{\text{inf}}[\frac{[\pi_{a'}]}{p}, \frac{p}{[\pi_{b'}]}]$ (where we regard both as subrings of $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$). Passing to p-adic completions, we obtain a map

$$\widehat{\mathbf{A}_{\text{inf}}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]} \to \widehat{\mathbf{A}_{\text{inf}}[\frac{[\pi_{a'}]}{p}, \frac{p}{[\pi_{b'}]}]}.$$

Inverting p, we get a ring homomorphism $B_{[a,b]} \to B_{[a',b']}$.

We can now define the object that we are interested in.

Definition 11. Let B denote the inverse limit $\varprojlim B_{[a,b]}$, taken over all closed intervals $[a, b] \subseteq (0, 1)$ such that a and b belong to the value group of $C^\flat$.