Lecture 27-Some Applications

December 7, 2018

In this final lecture, we describe some consequences of the classification of vector bundles on the Fargues-Fontaine curve. Recall that we have seen that the Fargues-Fontaine curve X behaves in many respects like an algebraic curve of genus 0. We now give one more heuristic piece of evidence for this.

Theorem 1. *The Fargues-Fontaine curve X is geometrically simply connected: that is, the projection map $X \to \mathrm{Spec}(\mathbf{Q}_p)$ induces an isomorphism of étale fundamental groups $\pi_1(X) \to \pi_1(\mathrm{Spec}(\mathbf{Q}_p)) = \mathrm{Gal}(\mathbf{Q}_p)$. Equivalently, pullback along the projection map*

$$X \to \mathrm{Spec}(\mathbf{Q}_p)$$

induces an equivalence of categories

$$\{ \textit{Étale covers of } \mathrm{Spec}(\mathbf{Q}_p)\} \to \{ \textit{Étale covers of } X\}.$$

We will need the following:

Lemma 2. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X which are semistable of slopes μ and μ', respectively. Then the tensor product $\mathcal{E} \otimes \mathcal{E}'$ is semistable (of slope $\mu + \mu'$).*

Proof. By the classification, we may assume without loss of generality that $\mathcal{E} = \rho_* \mathcal{O}_{X_E}(d)$, where $\rho : X_E \to X$ is the projection map for some finite extension E of $\mathbf{Q}_p$. Then $\mathcal{E} \otimes \mathcal{E}' = (\rho_* \mathcal{O}_{X_E}(d))\, \mathcal{E}' = \rho_*(\mathcal{O}_{X_E}(d) \otimes \rho^* \mathcal{E}')$. Since the functors ρ_* and ρ^* preserve semistability, we are reduced to proving that the tensor product functor $\mathcal{O}_{X_E}(d) \otimes \bullet$ preserves semistability, which is immediate from the definitions. □

Proof of Theorem 1. Let $\rho : \widetilde{X} \to X$ be a finite étale cover; we wish to show that $\widetilde{X}$ can be written uniquely as $X \times_{\mathrm{Spec}(\mathbf{Q}_p)} \mathrm{Spec}(E)$, where E is an étale $\mathbf{Q}_p$-algebra (that is, a product of finite extensions of $\mathbf{Q}_p$). Set $\mathcal{A} = \rho_* \mathcal{O}_{\widetilde{X}}$ and take $E = \mathrm{H}^0(X, \mathcal{A})$ (so that $E = \mathrm{H}^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is an algebra over $\mathbf{Q}_p$). To complete the proof, it will suffice to show the canonical map $E \otimes_{\mathbf{Q}_p} \mathcal{O}_X \to \mathcal{A}$ is an isomorphism (this forces $\widetilde{X} \simeq X \times_{\mathrm{Spec}(\mathbf{Q}_0)} \mathrm{Spec}(E)$, which in turn forces E to be an étale algebra over $\mathbf{Q}_p$). Equivalently, we wish to show that the vector bundle $\mathcal{A}$ is trivial.

By virtue of the classification, it will suffice to show that $\mathcal{A}$ is semistable of slope 0. We first observe that, since ρ is a finite étale morphism, the trace pairing $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \xrightarrow{\mathrm{tr}} \mathcal{O}_X$ is nondegenerate, so the vector bundle $\mathcal{A}$ is isomorphic to its dual $\mathcal{A}^\vee$. Since $\deg(\mathcal{A}) = -\deg(\mathcal{A}^\vee)$, it follows that $\deg(\mathcal{A}) = 0$ and therefore that $\mathcal{A}$ has slope zero. Assume, for a contradiction, that $\mathcal{A}$ is not semistable. Let $\mathcal{A}' \subseteq \mathcal{A}$ be the first step of the Harder-Narasimhan filtration of $\mathcal{A}$. Then $\mathcal{A}'$ is a semistable vector bundle of slope $\mu > 0$. Moreover, for every semistable vector bundle $\mathcal{E}$ of slope $> \mu$, every map $\mathcal{E} \to \mathcal{A}$ is zero. By Lemma 2, the tensor product $\mathcal{A}' \otimes \mathcal{A}'$ is semistable of slope $\mu + \mu > \mu$. It follows that the multiplication map

$$\mathcal{A}' \otimes \mathcal{A}' \hookrightarrow \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$$

must be the zero map. Let $U \subseteq X$ be an affine open subset over which the vector bundle $\mathcal{A}'$ has a nonzero section s. Then s can be viewed as a nonzero regular function on the scheme $\widetilde{X} \times_X U$ satisfying $s^2 = 0$. This is impossible, since the scheme $\widetilde{X} \times_X U$ is reduced (it is even a disjoint union of Dedekind schemes). □

Corollary 3. (1) *Pullback along the projection map $u : X \to \operatorname{Spec}(\mathbf{Q}_p)$ induces an equivalence of categories*

$$\{\textit{Finite abelian groups with a continuous action of } \operatorname{Gal}(\mathbf{Q}_p)\} \to \{\textit{Étale local systems on } X\}$$

(2) *If M is a finite abelian group with a continuous action of $\operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, then the induced map*

$$\mathrm{H}^*(\operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p); M) \to \mathrm{H}^*_{\mathrm{et}}(X, u^*M)$$

is an isomorphism for $* = 0, 1$.

Proof. Assertion (1) is immediate from Theorem 1. To prove (2), it will suffice to establish the following more general assertion:

(2′) Let E be a finite extension of $\mathbf{Q}_p$, let $u : X_E \to \operatorname{Spec}(E)$ be the projection map, and let M be a finite abelian group with a continuous action of $\operatorname{Gal}(\overline{\mathbf{Q}}_p/E)$, then the induced map

$$\mathrm{H}^*(\operatorname{Gal}(\overline{\mathbf{Q}}_p/E); M) \to \mathrm{H}^*_{\mathrm{et}}(X_E, u^*M)$$

is an isomorphism for $* = 0, 1$.

By a descent argument, to prove (2′) for a Galois module M, we are free to enlarge the field E. We may therefore assume without loss of generality that the Galois group $\operatorname{Gal}(\overline{\mathbf{Q}}_p/E)$ acts trivially on M. Then both cohomology groups are isomorphic to M when $* = 0$, and when $* = 1$ we are reduced to proving that u induces an isomorphism

$$\operatorname{Hom}(\pi_1(E), M) \to \operatorname{Hom}(\pi_1(X), M)$$

which follows from Theorem 1. □

Let C be a smooth proper algebraic curve defined over an algebraically closed field F of characteristic zero. Then the étale cohomology of C satisfies Poincaré duality. More precisely, for every positive integer n, there is a *trace map*

$$e_C : \mathrm{H}^2_{\mathrm{et}}(C; \mu_n) \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}$$

with the following property: for every constructible locally constant sheaf of $(\mathbf{Z}/n\mathbf{Z})$-modules $\mathcal{F}$ on C having dual $\mathcal{G} = \operatorname{Hom}(\mathcal{F}, \mu_n)$, the pairing

$$\mathrm{H}^*_{\mathrm{et}}(C; \mathcal{F}) \times \mathrm{H}^{2-*}_{\mathrm{et}}(C; \mathcal{G}) \to \mathrm{H}^2_{\mathrm{et}}(C; \mu_n) \xrightarrow{e_C} \mathbf{Z}/n\mathbf{Z}$$

is perfect.

Tate proved that the absolute Galois group $\operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is a (profinite) *Poincare duality* group of dimension 2. More precisely, there are isomorphisms $e_{\mathbf{Q}_p} : \mathrm{H}^2_{\mathrm{et}}(\operatorname{Spec} \mathbf{Q}_p; \mu_n) \xrightarrow{\sim} \mathbf{Z}/n\mathbf{Z}$ having the same property. In this sense, the scheme $\operatorname{Spec} \mathbf{Q}_p$ behaves like a smooth proper curve over an algebraically closed field. However, there are other respects in which it behaves differently. Note that on any $\mathbf{Z}[\frac{1}{n}]$-scheme Z, we have an exact sequence of étale sheaves

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 0.$$

This gives rise to a long exact sequence

$$\operatorname{Pic}(Z) \xrightarrow{n} \operatorname{Pic}(Z) \to \mathrm{H}^2_{\mathrm{et}}(Z; \mu_n) \to \operatorname{Br}(Z) \xrightarrow{n} \operatorname{Br}(Z)$$

(here $\mathrm{Br}(Z)$ denotes the *cohomological Brauer group* $\mathrm{H}^2_{\mathrm{et}}(Z; \mathbf{G}_m)$).

The Fargues-Fontaine curve provides a "geometric" way of thinking about Tate duality. Note that the projection map $X \to \mathrm{Spec}(\mathbf{Q}_p)$ induces a map of long exact sequences

$$\begin{array}{ccccccccc} \mathrm{Pic}(\mathbf{Q}_p) & \xrightarrow{n} & \mathrm{Pic}(\mathbf{Q}_p) & \longrightarrow & \mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Q}_p; \mu_n) & \longrightarrow & \mathrm{Br}(\mathbf{Q}_p) & \xrightarrow{n} & \mathrm{Br}(\mathbf{Q}_p) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Pic}(X) & \xrightarrow{n} & \mathrm{Pic}(X) & \longrightarrow & \mathrm{H}^2_{\mathrm{et}}(X; \mu_n) & \longrightarrow & \mathrm{Br}(X) & \xrightarrow{n} & \mathrm{Br}(X). \end{array}$$

One can show that the cohomology groups $\mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}(\mathbf{Q}_p); \mu_n)$ and $\mathrm{H}^2_{\mathrm{et}}(X; \mu_n)$ are both (canonically) isomorphic to $\mathbf{Z}/n\mathbf{Z}$, and that the middle vertical map is an isomorphism (combined with a descent argument, this implies that the comparison map of Corollary 3 is also an isomorphism on cohomology in degree 2). However, the "origin" of this $\mathbf{Z}/n\mathbf{Z}$ can be understood differently for $\mathrm{Spec}(\mathbf{Q}_p)$ and for X. The top line induces a short exact sequence

$$0 \to \mathrm{Pic}(\mathbf{Q}_p)/n\,\mathrm{Pic}(\mathbf{Q}_p) \to \mathrm{H}^2_{\mathrm{et}}(\mathrm{Spec}\,\mathbf{Q}_p; \mu_n) \to \mathrm{Br}(\mathbf{Q}_p)[n] \to 0$$

where the first term vanishes (since $\mathbf{Q}_p$ is a field) and the third term is isomorphic to $\mathbf{Z}/n\mathbf{Z}$ (since the Brauer group $\mathrm{Br}(\mathbf{Q}_p)$ is isomorphic to $\mathbf{Q}/\mathbf{Z}$ by the theory of the Hasse invariant). On the other hand, the lower line gives a short exact sequence

$$0 \to \mathrm{Pic}(X)/n\,\mathrm{Pic}(X) \to \mathrm{H}^2_{\mathrm{et}}(X; \mu_n) \to \mathrm{Br}(X)[n] \to 0$$

where the first term is isomorphic to $\mathbf{Z}/n\mathbf{Z}$ (since $\mathrm{Pic}(X) \simeq \mathbf{Z}$) and the third term vanishes by virtue of the following:

Theorem 4. *The Brauer group* $\mathrm{Br}(X)$ *is zero.*

Sketch. Let x be an element of the Brauer group of $\mathrm{Br}(X)$. Then x can be represented by an Azumaya algebra $\mathcal{A}$ on X, which we can regard as a vector bundle of rank n^2 on X for some $n > 0$. Let us assume that n has been chosen as small as possible. As in the proof of Theorem 1, the vector bundle $\mathcal{A}$ is isomorphic to its dual and therefore has slope 0. Assume that $\mathcal{A}$ is not semistable, and let $\mathcal{E} \subseteq \mathcal{A}$ be the first step of the Harder-Narasimhan stratification of $\mathcal{A}$. Let $\mathcal{I} \subseteq \mathcal{A}$ be the image of the multiplication map $\mathcal{A} \otimes \mathcal{E} \to \mathcal{A}$. Then $\mathcal{I}$ is a left ideal of $\mathcal{A}$, hence a left $\mathcal{A}$-module. It follows that $\mathcal{I}$ must have rank kn for some $k > 0$, and a local calculation shows that $k < n$. Let $\mathcal{B}$ denote the sheaf of endomorphisms of $\mathcal{I}$ as a left $\mathcal{A}$-module. One can then argue that $\mathcal{B}$ is an Azumaya algebra of rank k^2 and that the opposite algebra $\mathcal{B}^{\mathrm{op}}$ is Morita equivalent to $\mathcal{A}$ (via the bimodule $\mathcal{I}$), and therefore also represents the same Brauer class x. This contradicts the minimality of n. We may therefore assume that $\mathcal{A}$ is semistable of slope 0, and therefore of the form $A \otimes_{\mathbf{Q}_p} \mathcal{O}_X$ for some Azumaya algebra A over $\mathbf{Q}_p$. By minimality, it follows that A is a central division algebra over $\mathbf{Q}_p$; let $\mu = \frac{m}{n} \in \mathbf{Q}/\mathbf{Z} \simeq \mathrm{Br}(\mathbf{Q}_p)$ be its Hasse invariant. Then we can identify A with the algebra of endomorphisms of the isocrysal $V_{\frac{m}{n}}$ defined in the previous lecture. By functoriality, the algebra A acts (on the right) on the associated vector bundle $\mathcal{E}_{V_{\frac{m}{n}}} = \mathcal{O}(\frac{m}{n})$ on X: that is, we can regard $\mathcal{O}(\frac{m}{n})$ as a right $\mathcal{A} = A \otimes_{\mathbf{Q}_p} \mathcal{O}$-module. This induces an isomorphism $\mathcal{A} \simeq \mathrm{End}(\mathcal{O}(\frac{m}{n}))$, which shows that the Brauer class of $\mathcal{A}$ vanishes. □