Lecture 25-Functions on Y_E°

December 2, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let E be a finite extension of $\mathbf{Q}_p$, so that $\mathbf{Q}_p \subseteq E_0 \subseteq E$ where $\mathbf{Q}_p \hookrightarrow E_0$ is an unramified extension of degree d and $E_0 \hookrightarrow E$ is a totally ramified extension of degree e. We let $\mathbf{F}_q$ denote the residue field of E (so $q = p^d$) and π a uniformizer of the ring of integers $\mathcal{O}_E$. Fix an embedding $\mathbf{F}_q \hookrightarrow C^\flat$, which determines a map $\mathcal{O}_{E_0} \to \mathbf{A}_{\mathrm{inf}}$, hence $E_0 \to B$. Our first goal is to prove the following:

Proposition 1. *The ring $B \otimes_{E_0} E$ is an integral domain.*

Proof. We already know that B is an integral domain; let $K(B)$ denote its fraction field. We first note that B is integrally closed in $K(B)$. Given elements $f, g \in B$ such that $g \neq 0$ and $\frac{f}{g}$ is integral over B, it follows that for each point $y \in Y$ the image of $\frac{f}{g}$ in the field $B_{\mathrm{dR}}(y)$ is integral over $B^+_{\mathrm{dR}}(y)$. We therefore have $\frac{f}{g} \in B^+_{\mathrm{dR}}(y)$: that is, $\mathrm{ord}_y(f) \geq \mathrm{ord}_y(g)$. It follows that f is divisible by g in the ring B, so that $\frac{f}{g}$ belongs to B¿

Then $B \otimes_{E_0} E$ embeds in $K(B) \otimes_{E_0} E$. It will therefore suffice to show that the tensor product $K(B) \otimes_{E_0} E$ is a field. Enlarging E if necessary, we may assume that it is a Galois extension of E_0. Note that $K(B) \otimes_{E_0} E$ is an étale $K(B)$-algebra, so it is automatically a finite product of fields $K_1 \times \cdots \times K_m$. Then the subring $E \times \cdots \times E \subseteq K_1 \times \cdots \times K_m = K(B) \otimes_{E_0} E$ is invariant under the action of $\mathrm{Gal}(E/E_0)$, and can therefore be written as $L \otimes_{E_0} E$ for some E_0-subalgebra $L \subseteq K(B)$. Then L is an integral domain, so it is a field extension of E_0 (of degree m). We will complete the proof by showing that $L \otimes_{E_0} E$ is a field: that is, L and E are linearly disjoint over E_0. Since E is a totally ramified extension of E_0, it will suffice to show that L is an unramified extension of E_0. Let e' be the ramification degree of L over E_0 and let $f \in \mathcal{O}_L$ be a uniformizer. Then f is integral over $\mathcal{O}_{E_0} \subseteq B$, and therefore belongs to B (since B is integrally closed in $K(B)$). Note that, for each point $y \in Y$, we can identify $f(y)$ with the image of f under a continuous map of valued fields $L \to K_y$ (where K_y is the untilt of $C^\flat$ corresponding to y), and therefore have $|f(y)|_{K_y} = |p|_{K_y}^{1/e'}$. It follows that $|f|_\rho = \rho^{1/e'}$ for each $\rho \in (0,1)$: that is, we have $v_s(f) = \frac{s}{e'}$ for $s \in \mathbf{R}_{>0}$. Since the function $v_\bullet(f)$ is piecewise linear with *integer* slopes, we must have $e' = 1$ as desired. □

Corollary 2. *Let f be a nonzero element of $B \otimes_{E_0} E$ and let $N_{E/E_0}(f)$ denote its norm (along the finite flat map $\mathrm{Spec}(B) \to \mathrm{Spec}(B \otimes_{E_0} E)$). Then $N_{E/E_0}(f) \neq 0$.*

Corollary 3. *Let f and g be elements of $B \otimes_{E_0} E$, where $g \neq 0$. The following conditions are equivalent:*

(1) *The element f is divisible by g.*

(2) *For each point $\widetilde{y} \in Y_E^\circ$, we have $\mathrm{ord}_{\widetilde{y}}(f) \geq \mathrm{ord}_{\widetilde{y}}(g)$.*

Proof. The implication (1) ⇒ (2) is trivial. For the converse, assume that (2) is satisfied; we wish to show that g divides f. Write $N_{E/E_0}(g) = g \cdot h$. Replacing f by $f \cdot h$ and g by $g \cdot h = N_{E/E_0}(g)$, we can reduce to the case where g belongs to B. We can write f uniquely as a sum $f_0 + f_1\pi + \cdots + f_{e-1}\pi^{e-1}$, where each f_i belongs to B. It will therefore suffice to show that each f_i is divisible by g: that is, that we have $\mathrm{ord}_y(f_i) \geq \mathrm{ord}_y(g)$ for each point $y \in Y$. Let ξ be a uniformizer of the valuation ring $B^+_{\mathrm{dR}}(y)$ and set $n = \mathrm{ord}_y(g)$. Our hypothesis that $\mathrm{ord}_{\widetilde{y}}(f) \geq \mathrm{ord}_{\widetilde{y}}(g) = n$ for each point $\widetilde{y} \in Y^\circ_E$ lying over Y guarantees that the image of f vanishes in

$$\prod_{\widetilde{y} \to y} B^+_{\mathrm{dR}}(y)/(\xi^n) = (B^+_{\mathrm{dR}}(y)/(\xi^n)) \otimes_{E_0} E \simeq B^+_{\mathrm{dR}}(y)/(\xi^n) + \pi B^+_{\mathrm{dR}}(y)/(\xi^n) + \cdots + \pi^{e-1} B^+_{\mathrm{dR}}(y)/(\xi^n),$$

so that we have $\mathrm{ord}_y(f_i) \geq n$ for each i, as desired. □

Remark 4. Let f be a nonzero element of $B \otimes_{E_0} E$. Then the vanishing locus of f (as a subset of Y°_E) has at most countably many points. To see this, we observe that $N_{E/E_0}(f)$ is a nonzero element of B, and therefore vanishes on at most finitely many points of each annulus $Y_{[a,b]}$, and therefore at most countably many points of Y.

Corollary 5. *Let f be a nonzero element of $(B \otimes_{E_0} E)^{\varphi^d = \pi}$. Then the vanishing locus of f (as a subset of Y°_E) consists of a single orbit of $\varphi^{d\mathbf{Z}}$. Moreover, f has a simple zero at each point of this vanishing locus.*

Proof. Set $\pi' = N_{E/E_0}(\pi) \in \mathcal{O}_{E_0}$ and $f' = N_{E/E_0}(f)$, so that f belongs to $B^{\varphi^d = \pi'}$. Then the vanishing locus of f' is the image in Y of the vanishing locus of f, and the order of vanishing of f' at each point $y \in Y$ is the sum of the order of vanishing of f at each point of Y°_E lying over y. It will therefore suffice to show that f' vanishes on a single $\varphi^{d\mathbf{Z}}$-orbit (and with multiplicity 1 at each point of this orbit).

For each $\rho \in (0,1)$, we have

$$\rho^{p^d} \cdot |f'|_{\rho^{p^d}} = |\pi' f'|_{\rho^{p^d}} = |f'^{\varphi^d}|_{\rho^{p^d}} = |f'|_\rho^{p^d}$$

or, setting $s = -\log(\rho)$,

$$p^d s + v_{p^d s}(f') = p^d v_s(f).$$

Differentiating with respect to s and dividing by p^d, this gives

$$1 + \partial_- v_{p^d s}(f') = \partial_- v_s(f')$$

which implies that f' has exactly one zero (counted with multiplicity) on the half-open annulus $Y_{(\rho^p, \rho]} = \{y \in Y : \rho^p < d(y,0) \leq \rho\}$, Since the vanishing locus of f' is a union of $\varphi^{d\mathbf{Z}}$ orbits, it must consist of a single such orbit (with multiplicity 1). □

Let $\mathbf{G}_{\mathrm{LT}}$ denote the Lubin-Tate group of the previous lecture, and let $\sigma : \mathbf{G}_{\mathrm{LT}}(\mathcal{O}^\flat_C) \to \mathbf{G}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E)$ be the section of the projection map constructed in the previous lecture (which reduces to the usual Teichmüller section in the case $E = \mathbf{Q}_p$ and $\mathbf{G}_{\mathrm{LT}} = \widehat{\mathbf{G}}_m$).

Proposition 6. *The map*

$$\log_F(\sigma(\bullet)) : \mathbf{G}_{\mathrm{LT}}(\mathcal{O}^\flat_C) \to (B \otimes_{E_0} E)^{\varphi^d = \pi}$$

is an isomorphism.

Proof. We first prove injectivity. Let $\widetilde{y}$ be any point of Y°_E having image $y \in Y$ and let K_y denote the corresponding untilt of $C^\flat$, so that $\widetilde{y}$ equips $\mathcal{O}_{K_y}$ with the structure of a $\mathcal{O}_E$-algebra. Note that, on a sufficiently small ball around the origin in K_y, the construction $t \mapsto \log_F(t)$ is bijective (this happens already for the open ball of radius $|p|^{1/(p-1)}_{K_y}$). For any $t \in \mathbf{G}_{\mathrm{LT}}(K_y)$, the image $[\pi^m](t)$ will belong to this ball for $m \gg 0$. It follows that

$$(\log_F(t) = 0) \Leftrightarrow (\log_F([\pi^m](t)) = 0) \Leftrightarrow ([\pi^m](t) = 0 \text{ for } m \gg 0).$$

Let u be any nonzero element of $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$. The above reasoning shows that $\log_F(\sigma(u))$ vanishes at a point $\widetilde{y} \in Y_E^\circ$ if and only if $[\pi^m]\sigma(u) = \sigma(\varphi^{md}(u))$ vanishes at $\widetilde{y}$ for some $m \gg 0$. Since each $\sigma(\varphi^{md}(u))$ is a nonzero element of

$$\mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \simeq \mathfrak{m}_{\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E},$$

there are at most countably many points $\widetilde{y} \in Y_E^\circ$ which satisfy this condition (Remark 4). It follows that $\log_F(\sigma(u))$ cannot be identically zero.

We now prove surjectivity. Let f be a nonzero element of $(B \otimes_{E_0} E)^{\varphi^d = \pi}$. Then the vanishing locus of f consists of a single $\varphi^{d\mathbf{Z}}$ orbit on Y_E° (Corollary 5). In Lecture 24, we constructed a nonzero element $u \in \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$ such that $\log_F(\sigma(u))$ vanishes on the same locus. Since $\log_F(\sigma(u))$ is also nonzero (by the first part of the proof), it follows that $\lambda \log_F(\sigma(u)) = f$, where λ is a unit in the ring $B \otimes_{E_0} E$. It then follows that λ is a nonzero element of

$$(B \otimes_{E_0} E)^{\varphi^d = 1} \simeq (B \otimes_{\mathbf{Q}_p} E)^{\varphi = 1} \simeq B^{\varphi = 1} \otimes_{\mathbf{Q}_p} E \simeq E.$$

Replacing u by λu (using the structure of $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$ as a vector space over E), we can arrange that $\log_F(\sigma(u)) = f$. □

It follows from the preceding discussion that we have canonical bijections

$$\begin{aligned} \{\text{Closed points of } X_E\} &\simeq \{\varphi^{d\mathbf{Z}}\text{-orbits on } Y_E^\circ\} \\ &\simeq ((B \otimes_{E_0} E)^{\varphi^d = \pi} - \{0\})/E^\times \\ &\simeq (\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) - \{0\})/E^\times. \end{aligned}$$