Lecture 24-Lubin-Tate Formal Groups

November 30, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let E be a finite extension of $\mathbf{Q}_p$, so that $\mathbf{Q}_p \subseteq E_0 \subseteq E$ where $\mathbf{Q}_p \hookrightarrow E_0$ is an unramified extension of degree d and $E_0 \hookrightarrow E$ is a totally ramified extension of degree e. Set $q = p^d$. Let $\mathcal{O}_E$ denote the ring of integers of E and let $\pi \in \mathcal{O}_E$ be a uniformizer. Let $F(u,v)$ be the Lubin-Tate formal group law associated to the polynomial $f(t) = \pi t + t^q$ and let $\mathbf{G}_{\mathrm{LT}}$ denote the associated formal group. That is, F is the unique formal $\mathcal{O}_E$-module satisfying

$$[\pi](t) = \pi t + t^q.$$

In the previous lecture, we viewed the formal group $\mathbf{G}_{\mathrm{LT}}$ as a functor

$$\{\text{Commutative } \mathcal{O}_E\text{-algebras}\} \to \{\text{Abelian groups}\}$$

given by $\mathbf{G}_{\mathrm{LT}}(A) = \{$ Nilpotent elements of $A\}$ (with group structure given by $(u,v) \mapsto F(u,v)$). In this lecture, it will be convenient to consider a more general construction.

Notation 1. Let A be an $\mathcal{O}_E$-algebra which is complete with respect to an ideal I, which we view as a *topological* commutative ring by endowing it with the I-adic topology. We then define

$$\begin{aligned} \mathbf{G}_{\mathrm{LT}}(A) &= \varprojlim \mathbf{G}_{\mathrm{LT}}(A/I^n) \\ &= \{\text{Topologically nilpotent elements } x \in A\}. \end{aligned}$$

Beware that there is some ambiguity in our notation: the definition of $\mathbf{G}_{\mathrm{LT}}(A)$ depends on whether we view A as a discrete $\mathcal{O}_E$-algebra (in which case it consists only of nilpotent elements of A) or as a topological $\mathcal{O}_E$-algebra (in which case it consists of the topologically nilpotent elements of A; in particular, it includes the ideal I). Hopefully our usage will be clear in context.

Variant 2. *Let A be a $\mathcal{O}_E$-algebra which is complete with respect to an ideal I. We define $\widetilde{\mathbf{G}}_{\mathrm{LT}}(A)$ by the formula*

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) = \varprojlim(\cdots \xrightarrow{p} \mathbf{G}_{\mathrm{LT}}(A) \xrightarrow{p} \mathbf{G}_{\mathrm{LT}}(A).$$

We refer to the functor $A \mapsto \widetilde{\mathbf{G}}_{\mathrm{LT}}(A)$ as the universal cover of the Lubin-Tate formal group $\mathbf{G}_{\mathrm{LT}}$.

Remark 3. In the situation of Variant 2, the universal cover $\widetilde{\mathbf{G}}_{\mathrm{LT}}$ can also be defined as the inverse limit of the tower

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) = \varprojlim(\cdots \xrightarrow{\pi} \mathbf{G}_{\mathrm{LT}}(A) \xrightarrow{\pi} \mathbf{G}_{\mathrm{LT}}(A)$$

(since π^e is a unit multiple of p in the ring $\mathcal{O}_E$.

Example 4. Let K be an algebraically closed field containing E which is complete with respect to an absolute value $|\bullet|_K$ compatible with the valuation on $\mathcal{O}_E$. Then, as a set, we can identify $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$ with the maximal ideal $\mathfrak{m}_K$ of the valuation ring $\mathcal{O}_K$. Under this identification, multiplication by π is given by $t \mapsto \pi t + t^q$. Since K is algebraically closed, this map is surjection, and every element of $\mathfrak{m}_K$ has exactly q

preimages. In other words, the map $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K) \xrightarrow{\pi} \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$ is a surjection of $\mathcal{O}_E$-modules whose kernel has order q, and must therefore be isomorphic to the residue field $\mathcal{O}_E/(\pi) \simeq \mathbf{F}_q$ as a module over $\mathcal{O}_E$. It follows that the canonical map $\widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_K) \to \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$ is surjective.

We can be a bit more precise: for each $n \geq 0$, the kernel of the map $\pi^n : \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K) \to \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$ has order q^n, but contains only q elements which are annihilated by π. It follows that this kernel must be isomorphic to the quotient $\mathcal{O}_E/(\pi^n)$ as a module over $\mathcal{O}_E$. Passing to the inverse limit, we deduce that the kernel of the surjection

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_K) \twoheadrightarrow \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$$

is free of rank 1 as a $\mathcal{O}_E$-module.

Example 5. In the situation of Notation 1, suppose that $\pi \in \mathcal{O}_E$ vanishes in A (so that A has characteristic p, since π^e is a unit multiple of p). Then the polynomial $[\pi](t) = \pi t + t^q$ induces the Frobenius map

$$\mathbf{G}_{\mathrm{LT}}(A) \xrightarrow{x \mapsto x^q} \mathbf{G}_{\mathrm{LT}}(A).$$

Consequently, if A is a *perfect* $\mathcal{O}_E/(\pi)$-algebra, multiplication by π induces a bijection from $\mathbf{G}_{\mathrm{LT}}(A)$ to itself. It follows that multiplication by p also induces a bijection of $\mathbf{G}_{\mathrm{LT}}(A)$ with itself (again, because p is a unit multiple of π^e). In particular, the projection map $\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) \to \mathbf{G}_{\mathrm{LT}}(A)$ is a bijection.

Example 6. In the situation of Notation 1, assume that the ideal I contains p (this will be the case in all the situations we care about). In this case, the canonical map

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) \to \widetilde{\mathbf{G}}_{\mathrm{LT}}(A/I)$$

is bijective. To prove this, it will suffice to show that each of the maps

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A/I^{n+1}) \to \widetilde{\mathbf{G}}_{\mathrm{LT}}(A/I^n)$$

is an isomorphism. This follows from the observation that for $u, v \in I^n$, we have $F(u,v) \equiv u + v \pmod{I^{2n}}$, so that we have a short exact sequence of abelian groups

$$0 \to I^n/I^{n+1} \to \mathbf{G}_{\mathrm{LT}}(A/I^{n+1}) \to \mathbf{G}_{\mathrm{LT}}(A/I^n) \to 0$$

where the first group is annihilated by p; we therefore have a commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathbf{G}_{\mathrm{LT}}(A/I^{n+1}) & \xrightarrow{p} & \mathbf{G}_{\mathrm{LT}}(A/I^{n+1}) & \xrightarrow{p} & \mathbf{G}_{\mathrm{LT}}(A/I^{n+1}) \\
 & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
\cdots \longrightarrow & \mathbf{G}_{\mathrm{LT}}(A/I^{n}) & \xrightarrow{p} & \mathbf{G}_{\mathrm{LT}}(A/I^{n}) & \xrightarrow{p} & \mathbf{G}_{\mathrm{LT}}(A/I^{n}),
\end{array}$$

where the existence of the dotted arrows guarantees that the vertical maps induce an isomorphism after passing to the limit.

Remark 7. In the situation of Example 6, we do not need to divide out by the entire ideal I; if J is an ideal contained in I, then the map

$$\widetilde{\mathbf{G}}_{\mathrm{LT}}(A) \to \widetilde{\mathbf{G}}_{\mathrm{LT}}(A/J)$$

is an isomorphism (since both sides are isomorphic to $\widetilde{\mathbf{G}}_{\mathrm{LT}}(A/I)$.

Example 8. Consider the $\mathcal{O}_E$-algebra $\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E$. Note that the inclusion $E_0 \hookrightarrow E$ induces an isomorphism $\mathcal{O}_{E_0}/(p) \simeq \mathcal{O}_E/(\pi)$, and therefore also an isomorphism

$$\mathcal{O}_{C^\flat} \simeq \mathbf{A}_{\mathrm{inf}}/(p) \simeq (\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E)/(\pi).$$

We have a commutative diagram of abelian groups

$$\begin{array}{ccc} \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) & \xrightarrow{\sim} & \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \\ \downarrow & & \downarrow \sim \\ \mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) & \longrightarrow & \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat). \end{array}$$

Here the upper horizontal map is an isomorphism by Remark 7 and the right vertical map is an isomorphism by Example 5. It follows that the reduction map

$$\mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \to \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$$

is a surjection: in fact, it has a canonical section, given by the composition

$$\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \xleftarrow{\sim} \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \xleftarrow{\sim} \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \to \mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E).$$

In the case where $E = \mathbf{Q}_p$ and $\mathbf{G}_{\mathrm{LT}}$ is the formal multiplicative group, this is the Teichmüller section

$$(x \in 1 + \mathfrak{m}_C^\flat) \mapsto ([x] \in 1 + \mathfrak{m}_{\mathbf{A}_{\mathrm{inf}}}).$$

Example 9. Suppose we are given a point of Y_E°, corresponding to an untilt (K, ι) of $C^\flat$ equipped with an E_0-algebra map $E \to K$. We then have a canonical surjection $\mathbf{A}_{\mathrm{inf}} \to \mathcal{O}_K$, which induces a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) & \xrightarrow{\sim} & \widetilde{\mathbf{G}}_{\mathrm{LT}}(\mathcal{O}_K) \\ \downarrow & & \downarrow \\ \mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) & \longrightarrow & \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K). \end{array}$$

Here the top horizontal map is an isomorphism (by Remark 7 again) and the right vertical map is surjective (Example 4). We therefore obtain a surjection

$$\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \to \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K),$$

whose kernel is free of rank 1 as a $\mathcal{O}_E$-module.

In the special case where $E = \mathbf{Q}_p$ and $\mathbf{G}_{\mathrm{LT}}$ is the formal multiplicative group, this reduces to the map

$$1 + \mathfrak{m}_C^\flat \to 1 + \mathfrak{m}_K \qquad x \mapsto x^\sharp.$$

Construction 10. Let $\log_F$ denote the logarithm for the formal group law F, which we regard as a power series

$$\log_F(t) = t + \frac{c_2}{2}t^2 + \frac{c_3}{3}t^3 + \cdots \in E[[t]].$$

We observed in the previous lecture that the coefficients c_n belong to $\mathcal{O}_E$. Let x be any element belonging to the maximal ideal of the local ring $\mathbf{A}_{\mathrm{inf}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E$. We claim that the power series

$$\log_F(x) = x + \frac{c_2}{2}x^2 + \frac{c_3}{3}x^3 + \cdots$$

converges in the ring $B \otimes_{E_0} E = B + \pi B + \cdots + \pi^{e-1}B$. To prove this, we observe that each product $c_n x^n$ can be written uniquely as a sum

$$c_n x^n = a_{n,0} + a_{n,1}\pi + \cdots + a_{n,e-1}\pi^{e-1},$$

where each coefficient $a_{n,i}$ belongs to $\mathbf{A}_{\inf}$; we claim that the elements $\frac{1}{n}a_{n,i}$ converge to 0 (as $n \to \infty$) with respect to each of the Gauss norms $| \bullet |_\rho$ on $\mathbf{A}_{\inf}$. To prove this, write $x = x_0 + \pi y$, where x_0 belongs to the maximal ideal of $\mathbf{A}_{\inf}$. Then, for $n \geq e(m+1)$, the element $c_n x^n$ belongs to the ideal generated by the elements

$$x_0^{em}, px_0^{e(m-1)}, \cdots, p^{m-1}x_0^e, p^m \in \mathbf{A}_{\inf}.$$

It follows that each coefficient of $c_n x^n$ (when written as a sum of powers of π) has Gauss norm

$$\leq \max(|x_0^{em}|_\rho, |px_0^{e(m-1)}|_\rho, \cdots, |p^m|_\rho) = \max |x_0|_\rho^{em}, \rho^m.$$

These norms decay exponentially as $n \to \infty$, while the Gauss norms $|\frac{1}{n}|_\rho$ grow linearly in n.

Construction 10 supplies a canonical map $\mathcal{O}_E$-linear map

$$\mathbf{G}_{\mathrm{LT}}(\mathbf{A}_{\inf} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \xrightarrow{\log_F} B \otimes_{E_0} E,$$

which is also equivariant with respect to the Frobenius endomorphism φ^d. Composing this map with the "Teichmüller section" of Example 8, we obtain a homomorphism of $\mathcal{O}_E$-modules

$$\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \to B \otimes_{E_0} E$$

which is again equivariant with respect to the Frobenius φ^d. Since $\mathcal{O}_C^\flat$ is an algebra over $\mathcal{O}_E/(\pi)$, it follows from the construction of the Lubin-Tate formal group that the Frobenius map φ^d coincides with multiplication by π on the $\mathcal{O}_E$-module $\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat)$. It follows that the preceding construction gives a map

$$\log_F : \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \to (B \otimes_{E_0} E)^{\varphi^d = \pi}.$$

This proves a weak form of the result promised in Lecture 22:

Theorem 11. *Let x be a closed point of X_E, corresponding to a subset $S \subseteq Y_E^\circ$ which is an orbit for the action of $\varphi^{d\mathbf{Z}}$. Then there exists a nonzero element $f \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$ vanishing on S.*

Proof. Choose a point y of S, corresponding to an untilt (K, ι) of $C^\flat$ equipped with an E_0-algebra map $E \to K$. Then Example 9 implies that the natural map

$$\mathbf{G}_{\mathrm{LT}}(\mathcal{O}_C^\flat) \twoheadrightarrow \mathbf{G}_{\mathrm{LT}}(\mathcal{O}_K)$$

is a surjection, whose kernel is a free $\mathcal{O}_E$-module of rank 1. Let u be a generator of the kernel. We then take $f = \log_F(u) \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$. By construction, f vanishes at the point y (and therefore on the entire orbit S). □

To completely fulfill our promise from Lecture 22, we need to show that the function f of Theorem 11 vanishes simply at each point of the orbit S, and does not vanish on any other point of Y_E°. This is actually automatic: we will prove this in the next lecture.