Lecture 22: Line Bundles on Covers

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Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let X denote the Fargues-Fontaine curve, given by

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n}).$$

In the previous lecture, we explained a strategy for producing semistable vector bundles of any given slope $\lambda = \frac{m}{n}$ on X:

- First, choose a finite degree n extension E of $\mathbf{Q}_p$.
- Then, choose a line bundle $\mathcal{L}$ of degree m on the fiber product $X_E = X \times_{\mathrm{Spec}(\mathbf{Q}_p)} \mathrm{Spec}(E)$.

The direct image of $\mathcal{L}$ along the map $X_E \to X$ is then a semistable vector bundle of degree m and rank n. This vector bundle is *a priori* dependent on the choice of extension E and line bundle $\mathcal{L}$. But it turns out not to matter: up to isomorphism, the resulting vector bundle depends only on the integers d and n. First, independence of $\mathcal{L}$ is a consequence of the following:

Theorem 1. *Let E be a finite extension of $\mathbf{Q}_p$. Then the degree map* $\deg : \mathrm{Pic}(X_E) \to \mathbf{Z}$ *is an isomorphism.*

To prove Theorem 1, it will suffice to show that, for every pair of closed points $x, x' \in X_E$, we have $\mathcal{O}_{X_E}(x) = \mathcal{O}_{X_E}(x')$. We have already seen that this is true when $E = \mathbf{Q}_p$. Essentially, we proved this by observing that $\mathcal{O}_X(x)$ and $\mathcal{O}_X(x')$ can be identified with another line bundle $\mathcal{O}(1)$, whose definition did not depend on a choice of point of X. We would like to show that something similar happens for the scheme X_E.

Notation 2. For the remainder of this lecture, we fix a finite extension field E of $\mathbf{Q}_p$ of degree n. Then the inclusion $\mathbf{Q}_p \hookrightarrow E$ admits an essentially unique factorization as

$$\mathbf{Q}_p \hookrightarrow E_0 \hookrightarrow E,$$

where E_0 is an unramified extension of $\mathbf{Q}_p$ having some degree d (so that $E_0 \simeq W(\mathbf{F}_{p^d})[\frac{1}{p}]$ and E is a totally ramified extension of E_0 having some degree e; we then have $n = d \cdot e$. We let $\mathcal{O}_E$ denote the ring of integers of E, and $\pi \in \mathcal{O}_E$ a choice of uniformizer.

Exercise 3. Choose an embedding $\mathbf{F}_{p^d} \hookrightarrow \mathcal{O}_C^\flat$, which extends to a map $W(\mathbf{F}_{p^d}) \to W(\mathcal{O}_C^\flat) = \mathbf{A}_{\mathrm{inf}} \to B$ and therefore a map $E_0 \to B$, whose image is stable under the dth power of the Frobenius map. Let t be any homogeneous element of the graded ring $\bigoplus_{n \geq 0} B^{\varphi = p^n}$. Show that the canonical map

$$B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E \to B[\frac{1}{t}] \otimes_{E_0} E$$

induces an isomorphism

$$(B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E)^{\varphi=1} \simeq (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=1}.$$

In the special case $E_0 = E$, this recovers the isomorphism $(B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E)^{\varphi=1} \simeq B[\frac{1}{t}]^{\varphi^n=1}$ of the previous lecture.

It follows that, if $U \subseteq X$ is the complement of the vanishing locus of t and we set $U_E = U \times_{\mathrm{Spec}(\mathbf{Q}_p)} E$, then (when t has positive degree) we can write $U_E = \mathrm{Spec}((B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=1})$.

Construction 4. We attempt to construct a line bundle $\mathcal{O}_{X_E}(1)$ on X_E as follows:

- To each affine open subset $U \subseteq X$ as above (given by the complement of the vanishing locus of t), we set
$$\mathcal{O}_{X_E}(1)(U_E) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=\pi}.$$

To simultaneously show that this construction "works" and prove Theorem 1, it will suffice to show that $\mathcal{O}_{X_E}(1)$ agrees with the line bundle $\mathcal{O}_{X_E}(x)$, for any choice of point $x \in X_E$. In other words, we need to find a section of $\mathcal{O}_{X_E}(1)$ which vanishes exactly at the point x. First, we need some terminology.

Notation 5. In the previous lecture, we let Y_E denote the set of isomorphism classes of triples (K, ι, u), where (K, ι) is an untilt of $C^\flat$ and $u : E \to K$ is an embedding of fields. Let $Y_E^\circ \subseteq Y_E$ denote the subset consisting of those triples where $u|_{E_0}$ is given by the composite map $E_0 \to B \to K$ (corresponding to the embedding $\mathbf{F}_{p^d} \hookrightarrow C^\flat$ that we have chosen). Then Y_E° is not stable under the Frobenius, but is stable under its dth power; moreover, the inclusion $Y_E^\circ \hookrightarrow Y_E$ induces a bijection

$$Y_E^\circ/\varphi^{d\mathbf{Z}} \simeq Y_E/\varphi^{\mathbf{Z}}.$$

Recall that, for each point $y = (K, \iota)$ of Y, we have an evaluation map

$$B \to K \qquad f \mapsto f(y).$$

If we promote y to a point $\overline{y} = (K, \iota, e)$ of Y_E°, then this evaluation map admits an E-linear extension

$$B \otimes_{E_0} E \to K,$$

which we will denote by $f \mapsto f(\overline{y})$.

In fact, we can do a little bit better. Recall that K can be identified with the residue field of a discrete valuation ring $B_{\mathrm{dR}}^+(y)$ (with uniformizer we denote by ξ) and that the homomorphism $B \to K$ lifts to a map

$$B \to B_{\mathrm{dR}}^+(y) \qquad f \mapsto \widehat{f}_y.$$

In particular, this allows us to view $B_{\mathrm{dR}}^+(y)$ as an algebra over the field $E_0 \subseteq B$. Since E is a separable extension field of E_0, the E_0-algebra map

$$E \xrightarrow{u} K \simeq B_{\mathrm{dR}}^+(y)/(\xi)$$

lifts uniquely to a homomorphism $E \to B_{\mathrm{dR}}^+(y)$. Amalgamating, we obtain a homomorphism

$$B \otimes_{E_0} E \to B_{\mathrm{dR}}^+(y),$$

which we will denote by $f \mapsto \widehat{f}_{\overline{y}}$. In particular, this allows us to define an *order of vanishing* $\mathrm{ord}_{\overline{y}}(f) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ for each $f \in B \otimes_{E_0} E$: namely, the supremum of those integers m such that $\widehat{f}_{\overline{y}}$ is divisible by ξ^m.

We will deduce Theorem 1 from the following result, which we prove in the next lecture:

Theorem 6. *Let x be a closed point of X_E, corresponding to a subset $S \subseteq Y_E^{\circ}$ which is an orbit for the action of $\varphi^{d\mathbf{Z}}$. Then there exists an element $f \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$ satisfying*

$$\mathrm{ord}_{\overline{y}}(f) = \begin{cases} 1 & \text{if } \overline{y} \in S \\ 0 & \text{otherwise.} \end{cases}$$

Example 7. In the case $E = \mathbf{Q}_p$ and $\pi = p$, we can choose $f \in B^{\varphi = p}$ to be an element of the form $\log([\epsilon])$ for $\epsilon \in 1 + \mathfrak{m}_C^{\flat}$.

Proof of Theorem 1 from Theorem 6. Assuming Theorem 6, we show that for each closed point $x \in X_E$, the line bundle $\mathcal{O}_{X_E}(x)$ on X_E is isomorphic to the presheaf $\mathcal{O}_{X_E}(1)$ described in Construction 4: this will show both that $\mathcal{O}_{X_E}(1)$ extends to a line bundle and that $\mathcal{O}_{X_E}(x)$ is independent of x. Choose $f \in (B \otimes_{E_0} E)^{\varphi^d = \pi}$ satisfying the conclusion of Theorem 6. We will show that, for every affine open subset $U \subseteq X$ (complementary to the vanishing locus of some homogeneous element $t \in \bigoplus B^{\varphi = p^m}$), multiplication by f induces an isomorphism

$$\mathcal{O}_{X_E}(x)(U_E) \to \mathcal{O}_{X_E}(1)(U_E) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d = \pi}.$$

Note that $B \otimes_{E_0} E$ is a finite flat ring extension of B (of degree e). Let $N(f) \in B$ denote the norm of f along this ring extension (that is, the determinant of the B-module homomorphism of $B \otimes_{E_0} E$ given by multiplication by f). Note that, for each point $y \in Y$, we have $\widehat{N(f)}_y = \prod \widehat{f}_{\overline{y}}$, where the product is taken over the set of all preimages of y in Y_E°. It follows that the vanishing locus of $N(f)$ is given by a single orbit of $\varphi^{d\mathbf{Z}}$ on Y (and that $N(f)$ has simple zeros at each point where it vanishes). Then the product $N(f)\varphi(N(f))\varphi^2(N(f)) \cdots \varphi^{d-1}(N(f)) \in B$ vanishes on a single $\varphi^{\mathbf{Z}}$-orbit of Y (again with simple zeros), and can therefore be written as a product $u \log([\epsilon])$ where u is an invertible element of B and $\epsilon \in 1 + \mathfrak{m}_C^{\flat}$. Here $\log([\epsilon])$ vanishes at a single point of X, which can be identified with the image of x under the projection map $X_E \to X$. Note that since f divides the norm $N(f)$, it divides the product $N(f)\varphi(N(f))\varphi^2(N(f)) \cdots \varphi^{d-1}(N(f)) = u \log([\epsilon])$, and therefore also divides $\log([\epsilon])$.

We now distinguish two cases:

- Suppose that x does not belong to U_E. Then $\log([\epsilon])$ is a divisor of t, so f is a divisor of t and is therefore invertible in the ring $B[\frac{1}{t}] \otimes_{E_0} E$. In this case, multiplication by f induces an isomorphism of
$$\mathcal{O}_{X_E}(x)(U_E) = (B \otimes_{E_0} E)^{\varphi^d = 1} \xrightarrow{f} (B \otimes_{E_0} E)^{\varphi^d = \pi},$$
with inverse given by multiplication by $\frac{1}{f}$.

- Suppose that x belongs to U_E. Choose some other point $x' \in X_E$ which does not belong to U_E, and let $f' \in (B \otimes_{E_0} E)^{\varphi = \pi}$ satisfy the conclusion of Theorem 6 for the point x'. The preceding argument shows that f' is invertible in $B[\frac{1}{t}] \otimes_{E_0} E$. It follows that the ratio $\frac{f}{f'}$ is a well-defined element of $(B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi = 1}$, which we can identify with a regular function on U_E with a simple zero at the point x. Consequently, multiplication by $\frac{f'}{f}$ induces an isomorphism $\mathcal{O}_{X_E}(U_E) \to \mathcal{O}_{X_E}(x)(U_E)$. It will therefore suffice to show that the composite map
$$\mathcal{O}_{X_E}(U_E) \xrightarrow{\frac{f'}{f}} \mathcal{O}_{X_E}(x)(U_E) \xrightarrow{f} (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d = \pi}$$
is an isomorphism. In other words, we may replace x by x' and thereby reduce to the case treated above.

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