# Lecture 21: Covers of the Fargues-Fontaine Curve

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Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic $p$. Let $X$ denote the Fargues-Fontaine curve, given by

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n}).$$

Recall that if $\mathcal{E}$ is a nonzero vector bundle on $X$, the slope of $\mathcal{E}$ is defined by the formula

$$\mathrm{slope}(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

We say that $\mathcal{E}$ is *semistable* if every nonzero subbundle $\mathcal{E}' \subseteq \mathcal{E}$ satisfies $\mathrm{slope}(\mathcal{E}') \leq \mathrm{slope}(\mathcal{E})$. Our first goal in this lecture is to prove the following:

**Proposition 1.** *Let $\lambda = \frac{d}{n}$ be a rational number (where $d$ and $n$ are integers with $n > 0$). Then there exists a semistable vector bundle $\mathcal{E}$ on $X$ having degree $d$ and rank $n$ (hence slope $\lambda = \frac{d}{n}$).*

**Remark 2.** The vector bundle $\mathcal{E}$ appearing in the statement of Proposition 1 is unique up to isomorphism. We will return to this point in a future lecture.

In the case $n = 1$, Proposition 1 is trivial. Note that every line bundle $\mathcal{L}$ on $X$ is automatically semistable (since the only nonzero subbundle of $\mathcal{L}$ is $\mathcal{L}$ itself), so Proposition 1 merely asserts that for every integer $d$, there exists a line bundle of degree $d$. Here we are exploiting the fact that line bundles on $X$ are easy to make: every divisor $D \subseteq X$ determines a line bundle $\mathcal{O}_X(D)$ on $X$. To produce vector bundles, we will need to work harder. One possible strategy is to look for a map of schemes

$$\pi : \widetilde{X} \to X$$

which is finite and flat of degree $n$. In that case, for any line bundle $\mathcal{L}$ on $\widetilde{X}$, the direct image $\pi_*(\mathcal{L})$ will be a vector bundle of rank $n$ on $X$. There is an obvious source of examples to consider.

**Construction 3.** Let $E$ be a finite extension of the field $\mathbf{Q}_p$ having degree $n$. We let $X_E$ denote the fiber product $X \times_{\mathrm{Spec}(\mathbf{Q}_p)} \mathrm{Spec}(E)$.

Let us enumerate some easy properties of this construction.

- Since $E$ is a finite extension field of $\mathbf{Q}_p$, the map $\mathrm{Spec}(E) \to \mathrm{Spec}(\mathbf{Q}_p)$ is finite étale of degree $n$ (in particular, it is finite flat of degree $n$). It follows that the projection map $\pi : X_E \to X$ is finite étale of degree $n$. In particular, $X_E$ is also a Dedekind scheme.
- Since the unit map $\mathbf{Q}_p \to \mathrm{H}^0(X, \mathcal{O}_X)$ is an isomorphism, it follows that the unit map $E \to \mathrm{H}^0(X_E, \mathcal{O}_{X_E})$ is also an isomorphism. In particular, $\mathrm{H}^0(X_E, \mathcal{O}_{X_E})$ is a field, so the scheme $X_E$ is connected.

- Let $U \subseteq X$ be a nonempty affine open subset. Then $X - U$ can be given as the vanishing locus of a homogeneous element $t \in \bigoplus_{m \geq 0} B^{\varphi = p^m}$ (which is nonzero of positive degree). In this case, we have seen that $U$ can be described as the spectrum of the ring $B[\frac{1}{t}]^{\varphi=1}$. Setting $U_E = U \times_{\mathrm{Spec}(\mathbf{Q}_p)} E$, it follows that $U_E$ can be described as the spectrum of the ring

$$B[\frac{1}{t}]^{\varphi=1} \otimes_{\mathbf{Q}_p} E \simeq (B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E)^{\varphi=1}.$$

  Here we extend the Frobenius automorphism $\varphi : B \to B$ to the tensor product $B \otimes_{\mathbf{Q}_p} E$ by letting it act trivially on the second factor.

- Let $x \in X$ be a closed point corresponding to an untilt $K$ of $C^\flat$. Then the fiber product $X_E \times_X \{x\}$ can be identified with the spectrum of the tensor product $E \otimes_{\mathbf{Q}_p} K$. Since $K$ is algebraically closed, this tensor product just factors as a Cartesian product of $n$ copies of $K$. That is, every closed point of $X$ has exactly $n$ points of $X_E$ lying over it, each of which has the same residue field.

- Let $Y$ denote the set of isomorphism classes of characteristic zero untilts $(K, \iota)$ of $C^\flat$. Recall that the set of closed points of $X$ can be identified with the quotient $Y/\varphi^{\mathbf{Z}}$. Using the same reasoning, we see that the collection of closed points of the curve $X_E$ can be identified with the quotient $Y_E/\varphi^{\mathbf{Z}}$; here $Y_E$ denotes the set of isomorphism classes of triples $(K, \iota, u)$, where $K$ is a perfectoid field of characteristic zero, $\iota : C^\flat \simeq K^\flat$ is an isomorphism, and $u : E \to K$ is a map of $\mathbf{Q}_p$-algebras (note in this situation, $K$ is an algebraically closed extension field of $\mathbf{Q}_p$, so there are exactly $n$ choices for the embedding $u$).

**Example 4.** Suppose that $E$ is the unramified degree $n$ extension of $\mathbf{Q}_p$, given by $E = W(\mathbf{F}_{p^n})[\frac{1}{p}]$. If $K$ is an untilt of $C^\flat$, then the following data are equivalent:

- $\mathbf{Q}_p$-algebra maps $e : E \to K$.
- $\mathbf{Z}_p$-algebra maps $W(\mathbf{F}_{p^n}) \to \mathcal{O}_K$.
- $\mathbf{F}_p$-algebra maps $\mathbf{F}_{p^n} \to \mathcal{O}_K / p\, \mathcal{O}_K$.
- $\mathbf{F}_p$-algebra maps $\mathbf{F}_{p^n} \to \mathcal{O}_C^\flat / \pi$, where $\pi \in C^\flat$ satisfies $|\pi|_{C^\flat} = |p|_K$.
- $\mathbf{F}_p$-algebra maps $\mathbf{F}_{p^n} \to \mathcal{O}_C^\flat$ (since $\mathbf{F}_p$ is étale over $\mathbf{F}_p$).
- $\mathbf{F}_p$-algebra maps $\mathbf{F}_{p^n} \to C^\flat$.

We therefore obtain a bijection $Y_E \simeq Y \times \mathrm{Hom}_{\mathbf{F}_p}(\mathbf{F}_{p^n}, C^\flat)$; here $\mathrm{Hom}_{\mathbf{F}_p}(\mathbf{F}_{p^n}, C^\flat$ denotes the set of $\mathbf{F}_p$-algebra maps from $\mathbf{F}_{p^n}$ into $C^\flat$. This has exactly $n$ elements, which are cyclically permuted by the action of the Frobenius map $\varphi_C^\flat$. It follows that in this case, we have canonical bijections

$$\begin{aligned} \text{Closed points of } X_E &\simeq Y_E/\varphi^{\mathbf{Z}} \\ &\simeq (Y \times \mathrm{Hom}_{\mathbf{F}_p}(\mathbf{F}_{p^n}, C^\flat))/\varphi^{\mathbf{Z}} \\ &\simeq Y/\varphi^{n\mathbf{Z}}. \end{aligned}$$

In the situation of Example 4, the description of the set of closed points of $X_E$ has a counterpart at the level of functions. Let $U \subseteq X$ be a nonempty affine open subset and let $U_E \subseteq X_E$ be its inverse image in $X_E$, so that we can write

$$U_E = \mathrm{Spec}((B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E)^{\varphi=1}).$$

In the case where $E$ is unramified over $\mathbf{Q}_p$, this description can be simplified. Note that each embedding of $u : \mathbf{F}_{p^n} \hookrightarrow C^\flat$ factors through $\mathcal{O}_C^\flat$, and therefore induces a map

$$W(\mathbf{F}_{p^n}) \to W(\mathcal{O}_C^\flat) = \mathbf{A}_{\mathrm{inf}} \to B \to B[\frac{1}{t}],$$

which extends to a $\mathbf{Q}_p$-algebra map $\overline{u} : E \to B[\frac{1}{t}]$. Tensoring $\overline{u}$ with the identity map on $B[\frac{1}{t}]$, we obtain a map

$$q_u : B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E \to B[\frac{1}{t}].$$

**Exercise 5.** Show that the maps $q_u$ induce an isomorphism

$$B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E \simeq \prod_{u:\mathbf{F}_{p^n} \hookrightarrow C^\flat} B[\frac{1}{t}].$$

This doesn't require knowing much about the situation: you can replace $B[\frac{1}{t}]$ by any $\mathbf{Q}_p$-algebra $R$ which admits a map $E \to R$.

The Frobenius automorphism of $B[\frac{1}{t}]$ extends uniquely to an $E$-linear automorphism of $B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E$. Under the isomorphism

$$B[\frac{1}{t}] \otimes_{\mathbf{Q}_p} E \simeq \prod_{u:\mathbf{F}_{p^n} \hookrightarrow C^\flat} B[\frac{1}{t}],$$

this automorphism cyclically permutes the factors. Concretely, if we identify an element of the right hand side with an $n$-tuple $(f_0, f_1, \ldots, f_{n-1})$ of elements of $B[\frac{1}{t}]$, then we have $\varphi(f_0, f_1, \ldots, f_{n-1}) = (\varphi(f_{n-1}), \varphi(f_0), \varphi(f_1), \ldots, \varphi(f_{n-2}))$. It follows that $(f_1, f_2, \ldots, f_n)$ is invariant under the Frobenius if and only if we have $f_i = \varphi(f_{i-1})$ for $0 < i < n$ and $f_0 = \varphi(f_{n-1})$. The first equation guarantees that $f_i = \varphi^i(f_0)$ for $0 < i < n$, so that we can rewrite the second as $\varphi^n(f_0) = f_0$. This proves the following:

**Proposition 6.** *Let $E$ be the unramified degree $n$ extension of $\mathbf{Q}_p$ and let $U \subsetneq X$ be the vanishing locus of a homogeneous element $t \in \bigoplus B^{\varphi=p^m}$. Then we have*

$$U_E = \mathrm{Spec}(B[\frac{1}{t}]^{\varphi^n=1}).$$

Let us now return to the case where $E$ is *any* finite extension field of $\mathbf{Q}_p$. Let $\pi : X_E \to X$ be the projection map. Note that if $\mathcal{E}$ is any vector bundle of rank $r$ on $X_E$, then $\pi_* \mathcal{E}$ is a vector bundle of rank $nr$ on $X$. Moreover, this construction induces an equivalence of categories

$$\{\text{Vector bundles on } X_E\} \simeq \{\text{Vector bundles on } X \text{ with an action of the field } E\}.$$

Every vector bundle $\mathcal{E}$ on $X_E$ has a well-defined degree, defined by the formula $\deg(\mathcal{E}) = \deg(\pi_* \mathcal{E})$. If $\mathcal{E}$ is not zero, we can define the slope $\mathrm{slope}(\mathcal{E})$ by the formula $\mathrm{slope}(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})} = \frac{1}{n} \mathrm{slope}(\pi_* \mathcal{E})$.

**Proposition 7.** *Let $\mathcal{E}$ be a nonzero vector bundle on $X_E$ and let $\lambda$ be a rational number. The following conditions are equivalent:*

- $(a)$ *The vector bundle $\mathcal{E}$ is semistable of slope $\lambda$. That is,* $\mathrm{slope}(\mathcal{E}) = \lambda$ *and, for every nonzero subbundle $\mathcal{E}' \subseteq \mathcal{E}$, we have* $\mathrm{slope}(\mathcal{E}') \leq \lambda$.
- $(b)$ *The direct image $\pi_* \mathcal{E} \in \mathrm{Vect}(X)$ is semistable of slope $\frac{\lambda}{n}$, in the sense of the previous lecture.*

*Proof.* We have already observed that $\mathcal{E}$ has slope $\lambda$ if and only if $\pi_* \mathcal{E}$ has slope $\frac{\lambda}{n}$. The implication $(b) \Rightarrow (a)$ is now clear: note that $\pi_* \mathcal{E}$ is semistable and $\mathcal{E}'$ is a nonzero subbundle of $\mathcal{E}$, then $\pi_* \mathcal{E}'$ is a nonzero subbundle of $\pi_* \mathcal{E}$, and therefore satisfies

$$\deg(\mathcal{E}') = n \deg(\pi_* \mathcal{E}') \leq n \deg(\pi_* \mathcal{E}) = \deg(\mathcal{E}).$$

Conversely, suppose that $\mathcal{E}$ is semistable of slope $\lambda$.

Assume that $\mathcal{F} = \pi_* \mathcal{E}$ is not semistable; we will see that this leads to a contradiction. Let

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_m = \mathcal{F}$$

be the Harder-Narasimhan filtration of $\mathcal{F}$, with slopes $\lambda_i = \mathrm{slope}(\mathcal{F}_i / \mathcal{F}_{i-1})$ satisfying $\lambda_1 > \lambda_2 > \cdots > \lambda_m$. Then we must have $\lambda_1 > \mathrm{slope}(\mathcal{F}) = \frac{\lambda}{n}$. Note that, for any nonzero element $x \in E$, multiplication by $x$ induces an automorphism of $\mathcal{F} = \pi_* \mathcal{E}$ which automatically preserves the Harder-Narasimhan filtration. It follows that the action of $E$ on $\mathcal{F}$ preserves the subbundle $\mathcal{F}_1$, so that we can write $\mathcal{F}_1 = \pi_* \mathcal{E}'$ for some subbundle $\mathcal{E}' \subseteq \mathcal{E}$. We then have $\mathrm{slope}(\mathcal{E}') = n\lambda_1 > \lambda$, contradicting the semistability of $\mathcal{E}$. □

*Proof of Proposition 1.* Let $\lambda = \frac{d}{n}$ be a rational number; we wish to show that there exists a semistable vector bundle on $X$ having rank $n$ and degree $d$. Let $E$ be a finite extension of $\mathbf{Q}_p$ of degree $n$, and let $\mathcal{L}$ be a line bundle of degree $d$ on $X_E$ (for example, we can take $\mathcal{L} = \mathcal{O}_{X_E}(D)$, where $D$ is a divisor of degree $d$ on $X_E$). Then $\mathcal{L}$ is automatically semistable as a vector bundle on $X_E$ (since it has rank 1). Applying Proposition 7, we see that $\pi_* calL$ is a semistable vector bundle on $X$, which evidently has rank $n$ and degree $d$. □

Our next goal is to better understand the vector bundle produced by our proof of Proposition 1. *A priori*, it depends on a few choices: the finite extension $E \supseteq \mathbf{Q}_p$, and the choice of line bundle $\mathcal{L}$ on $X_E$. But it turns out that both of these choices are irrelevant. In the second case, this is because of the following:

**Theorem 8.** *Let $E$ be a finite extension of $\mathbf{Q}_p$. Then the degree map $\deg : \mathrm{Pic}(X_E) \to \mathbf{Z}$ is an isomorphism.*

In Lecture 19, we proved Theorem 8 in the special case $E = \mathbf{Q}_p$. The point was that for every pair of closed points $x, x' \in X$, the divisor $x - x'$ is linearly equivalent to zero: that is, we can find a rational function $f$ on $X$ having a simple zero at $x$ and a simple pole at $x'$. More precisely, we can take

$$f = \frac{\log([\epsilon])}{\log([\epsilon'])} \in B[\frac{1}{\log([\epsilon'])}]^{\varphi=1},$$

where $\epsilon, \epsilon' \in 1 + \mathfrak{m}_C^\flat$ are chosen so that the vanishing loci of $\log([\epsilon])$ and $\log([\epsilon'])$ in $Y$ are exactly the Frobenius orbits corresponding to the points $x$ and $x'$, respectively. The function $f$ is Frobenius-invariant, but individually the numerator and denominator are not: they both belong to the eigenspace $B^{\varphi=p}$. To prove Theorem 8 in general, we need to do something analogous for the cover $X_E$. We will return to this in the next lecture.