Lecture 20: The Harder-Narasimhan Filtration

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Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let X denote the Fargues-Fontaine curve, given by

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n}).$$

Our goal in this lecture is to show that every vector bundle $\mathcal{E}$ on X admits a canonical *Harder-Narasimhan filtration* (just as if X were an algebraic curve defined over a field).

We begin with some generalities. Recall that, if $\mathcal{E}$ is a nonzero vector bundle on X, the *slope* $\mathrm{slope}(\mathcal{E})$ is defined by the formula

$$\mathrm{slope}(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

Exercise 1. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence of nonzero vector bundles on X, so that we have equalities

$$\deg(\mathcal{E}) = \deg(\mathcal{E}') + \deg(\mathcal{E}'') \qquad \mathrm{rank}(\mathcal{E}) = \mathrm{rank}(\mathcal{E}') + \mathrm{rank}(\mathcal{E}'').$$

Using this, show that:

- If $\mathrm{slope}(\mathcal{E}') = \mathrm{slope}(\mathcal{E}'')$, then $\mathrm{slope}(\mathcal{E}') = \mathrm{slope}(\mathcal{E}) = \mathrm{slope}(\mathcal{E}'')$.
- If $\mathrm{slope}(\mathcal{E}') < \mathrm{slope}(\mathcal{E}'')$, then $\mathrm{slope}(\mathcal{E}') < \mathrm{slope}(\mathcal{E}) < \mathrm{slope}(\mathcal{E}'')$.
- If $\mathrm{slope}(\mathcal{E}') > \mathrm{slope}(\mathcal{E})$, then $\mathrm{slope}(\mathcal{E}') > \mathrm{slope}(\mathcal{E}) > \mathrm{slope}(\mathcal{E}'')$.

Remark 2. Let $\mathcal{E}$ be a vector bundle on X and let $\mathcal{E}' \subsetneq \mathcal{E}$ be a subsheaf which is a vector bundle of the same rank (so that the quotient $\mathcal{E}'' = \mathcal{E} / \mathcal{E}'$ is a coherent sheaf with finite support on X). Then $\deg(\mathcal{E}') < \deg(\mathcal{E})$ and therefore $\mathrm{slope}(\mathcal{E}') < \mathrm{slope}(\mathcal{E})$. To prove this, we can replace $\mathcal{E}$ and $\mathcal{E}'$ by their top exterior powers and thereby reduce to the case where $\mathcal{E}$ and $\mathcal{E}'$ are line bundles, in which case the result is obvious (since there are no nonzero maps from $\mathcal{O}(m)$ to $\mathcal{O}(n)$ for $m > n$, and every nonzero map from $\mathcal{O}(n)$ to itself is an isomorphism). Note that this can be regarded as a degenerate version of Exercise 1, where we adopt the convention that $\mathrm{slope}(\mathcal{E}'') = \infty$.

Definition 3. Let $\mathcal{E}$ be a nonzero vector bundle on X and let λ be a rational number. We say that $\mathcal{E}$ is *semistable of slope* λ if $\mathrm{slope}(\mathcal{E}) = \lambda$ and, for every nonzero subbundle $\mathcal{E}' \subseteq \mathcal{E}$, we have $\mathrm{slope}(\mathcal{E}') \leq \lambda$. By convention, we say that the zero vector bundle is semistable of every slope.

Remark 4. Let $\mathcal{E}$ be a vector bundle on X which is semistable of slope λ and let $\mathcal{E}' \subseteq \mathcal{E}$ be a coherent subsheaf. Then $\mathcal{E}'$ is also a vector bundle, but not necessarily a vector subbundle (since the quotient $\mathcal{E} / \mathcal{E}'$ might not be a vector bundle). However, $\mathcal{E}'$ is always contained in a vector subbundle $\overline{\mathcal{E}}' \subseteq \mathcal{E}$ of the same rank. Using Remark 2 we obtain

$$\mathrm{slope}(\mathcal{E}') \leq \mathrm{slope}(\overline{\mathcal{E}}') \leq \lambda.$$

Moreover, the first inequality is strict if $\mathcal{E}'$ is not a subbundle of $\mathcal{E}$.

Proposition 5. *Let $\mathcal{E}$ be a vector bundle on X which is semistable of rank λ. For any surjection of vector bundles $\mathcal{E} \twoheadrightarrow \mathcal{E}''$, we have* $\mathrm{slope}(\mathcal{E}'') \geq \lambda$.

Proof. We have an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, and the semistability of $\mathcal{E}$ gives $\mathrm{slope}(\mathcal{E}') \leq \mathrm{slope}(\mathcal{E}) = \lambda$. Applying Exercise 1, we see that $\mathrm{slope}(\mathcal{E}'') \geq \lambda$. □

Corollary 6. *Let $\mathcal{E}$ and $\mathcal{F}$ be semistable vector bundles of slopes λ and μ. If $\lambda > \mu$, then every map of vector bundles $f : \mathcal{E} \to \mathcal{F}$ vanishes.*

Proof. If $f \neq 0$, then the image $\mathrm{Im}(f)$ is a nonzero coherent subsheaf of $\mathcal{F}$, hence a vector bundle of rank > 0. Remark 4 and Proposition 5 then give

$$\lambda = \mathrm{slope}(\mathcal{E}) \leq \mathrm{slope}(\mathrm{Im}(f)) \leq \mathrm{slope}(\mathcal{F}) = \mu,$$

contradicting our assumption that $\lambda > \mu$. □

Proposition 7. *Let $f : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles on $\mathcal{E}$ which are semistable of slope λ. Then* $\ker(f)$ *and* $\mathrm{coker}(f)$ *(formed in the category of coherent sheaf on X) are vector bundles.*

Proof. If $f = 0$, there is nothing to prove. Otherwise, we again have inequalities

$$\lambda = \mathrm{slope}(\mathcal{E}) \leq \mathrm{slope}(\mathrm{Im}(f)) \leq \mathrm{slope}(\mathcal{F}) = \lambda.$$

It follows that equality must hold in both cases, so that $\mathrm{Im}(f)$ has slope λ. Moreover, Remark 4 shows that it is a vector subbundle of $\mathcal{F}$, so that $\mathrm{coker}(f)$ is a vector bundle on X and we have exact sequences

$$0 \to \ker(f) \to \mathcal{E} \to \mathrm{Im}(f) \to 0$$

$$0 \to \mathrm{Im}(f) \to \mathcal{F} \to \mathrm{coker}(f) \to 0.$$

Applying Exercise 1, we conclude that $\ker(f)$ and $\mathrm{coker}(f)$ (if nonzero) also have slope λ. Every subbundle of $\ker(f)$ can also be regarded as a subbundle of $\mathcal{E}$, and therefore has slope $\leq \lambda$ by virtue of our assumption that $\mathcal{E}$ is semistable. This proves the $\ker(f)$ is semistable of slope λ. We claim that $\mathrm{coker}(f)$ is also semistable of slope λ. Assume otherwise: then there exists a subbundle $\overline{\mathcal{F}}' \subseteq \mathrm{coker}(f)$ of slope $> \lambda$. Let $\mathcal{F}'$ be the inverse image of $\overline{\mathcal{F}}'$ in $\mathcal{F}$, so that we have an exact sequence

$$0 \to \mathrm{Im}(f) \to \mathcal{F}' \to \overline{\mathcal{F}}' \to 0.$$

Applying Exercise 1, we deduce that $\mathrm{slope}(\mathcal{F}') > \lambda$, contradicting the semistability of $\mathcal{F}$. □

Proposition 8. *Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of vector bundles on X. If $\mathcal{E}'$ and $\mathcal{E}''$ are semistable of slope λ, then so is $\mathcal{E}$.*

Proof. Exercise 1 shows that $\mathcal{E}$ has slope λ. Let $\mathcal{F} \subseteq \mathcal{E}$ be any vector subbundle. Let $\mathcal{F}' = \mathcal{F} \cap \mathcal{E}'$ and let $\mathcal{F}''$ be the image of $\mathcal{F}$ in $\mathcal{E}''$. Then $\mathcal{F}'$ and $\mathcal{F}''$ are vector bundles which can be regarded as subsheaves of $\mathcal{E}'$ and $\mathcal{E}''$, respectively, so Remark 4 implies that $\mathrm{slope}(\mathcal{F}'), \mathrm{slope}(\mathcal{F}'') \leq \lambda$. Using the exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

we deduce that $\mathrm{slope}(\mathcal{F}) \leq \lambda$. □

Corollary 9. *Let* $\mathrm{Coh}(X)$ *denote the category of coherent sheaves on X and let* $\mathrm{Vect}_\lambda(X) \subseteq \mathrm{Coh}(X)$ *denote the full subcategory whose objects are vector bundles on X which are semistable of slope* 0. *Then* $\mathrm{Vect}_\lambda(X)$ *is closed under kernels, cokernels, and extensions in* $\mathrm{Coh}(X)$. *In particular, it is an abelian category.*

Warning 10. The collection of *all* vector bundles on X does not form an abelian category (note that if $f : \mathcal{E} \to \mathcal{F}$ is a map of vector bundles, then in general the cokernel $\operatorname{coker}(f)$ in the category of coherent sheaves is not a vector bundle).

Definition 11. Let $\mathcal{E}$ be a vector bundle on X. We say that a filtration

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subsetneq \cdots \subseteq \mathcal{E}_m = \mathcal{E}$$

is a *Harder-Narasimhan filtration* if the following conditions are satisfied:

- Each of the quotient vector bundles $\mathcal{E}_i / \mathcal{E}_{i-1}$ is semistable of some slope λ_i.
- The slopes λ_i are strictly decreasing: that is, we have $\lambda_1 > \lambda_2 > \cdots > \lambda_m$.

Theorem 12. *Let $\mathcal{E}$ be a vector bundle on X. Then $\mathcal{E}$ has a unique Harder-Narasimhan filtration.*

Let us first establish uniqueness. We will proceed by induction on the rank r of $\mathcal{E}$. Suppose that $\mathcal{E}$ is equipped with two Harder-Narasimhan filtrations

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subsetneq \cdots \subseteq \mathcal{E}_m = \mathcal{E}$$

$$0 = \mathcal{E}'_0 \subsetneq \mathcal{E}'_1 \subseteq \mathcal{E}'_2 \subsetneq \cdots \subseteq \mathcal{E}'_n = \mathcal{E}\,.$$

where the successive quotients have slopes $\lambda_1 > \cdots > \lambda_m$ and $\lambda'_1 > \cdots > \lambda'_n$, respectively. We wish to show that these filtrations are the same. We will show that $\mathcal{E}_1 = \mathcal{E}'_1$; the desired result will then follow by applying the inductive hypothesis to the filtrations

$$0 = \mathcal{E}_1 / \mathcal{E}_1 \subseteq \mathcal{E}_2 / \mathcal{E}_1 \subsetneq \cdots \subseteq \mathcal{E}_m / \mathcal{E}_1 = \mathcal{E} / \mathcal{E}_1$$

$$0 = \mathcal{E}'_1 / \mathcal{E}'_1 \subseteq \mathcal{E}'_2 / \mathcal{E}'_1 \subsetneq \cdots \subseteq \mathcal{E}'_n / \mathcal{E}'_1 = \mathcal{E} / \mathcal{E}'_1\,.$$

We first claim that $\lambda_1 = \lambda'_1$. Suppose otherwise. Then we may assume without loss of generality that $\lambda_1 > \lambda'_1$. It follows that $\lambda_1 > \lambda'_i$ for $1 \leq i \leq n$. Applying Corollary 6, we conclude that $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}'_i / \mathcal{E}'_{i-1}) = 0$. Since $\mathcal{E}$ admits a finite filtration whose successive quotients are $\mathcal{E}'_i / \mathcal{E}'_{i-1}$, it follows that $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}) = 0$. This is a contradiction, since the inclusion map $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ is a nonzero element of $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E})$.

The equality $\lambda_1 = \lambda'_1$ guarantees that we have a strict inequality $\lambda_1 > \lambda'_i$ for $i > 1$. As above, we conclude that $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}'_i / \mathcal{E}'_{i-1}) = 0$. Since the quotient bundle $\mathcal{E} / \mathcal{E}'_1$ admits a finite filtration whose successive quotients have the form $\mathcal{E}'_i / \mathcal{E}'_{i-1}$ with $i > 1$, it follows that $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E} / \mathcal{E}'_1)$ vanishes. In particular, the composite map

$$\mathcal{E}_1 \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E} / \mathcal{E}'_1$$

must be zero, so we must have $\mathcal{E}_1 \subseteq \mathcal{E}'_1$. Applying the same argument with the roles of $\mathcal{E}_1$ and $\mathcal{E}'_1$ reversed, we deduce that $\mathcal{E}'_1 \subseteq \mathcal{E}_1$. We therefore have equality $\mathcal{E}_1 = \mathcal{E}'_1$, which (together with our inductive hypothesis) proves the uniqueness part of Theorem 12. To prove existence, we need the following:

Lemma 13. *Let $\mathcal{E}$ be a vector bundle on X. Then there exists an integer $N(\mathcal{E})$ with the following property: for every coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$, we have $\deg(\mathcal{F}) \leq N(\mathcal{E})$.*

Proof. We proceed by induction on the rank of $\mathcal{E}$. Note that if $\mathcal{E}$ is a line bundle, then every subsheaf $\mathcal{F} \subseteq \mathcal{E}$ is either a line bundle of smaller degree or zero; we can therefore take $N(\mathcal{E}) = \max(\deg(\mathcal{E}), 0)$. To handle the general case, we observe that if $\mathcal{E}$ has rank > 1 then we can choose an exact sequence of vector bundles

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

where $\mathcal{E}'$ and $\mathcal{E}''$ have smaller rank (for example, we can take $\mathcal{E}'$ to be the line subbundle of $\mathcal{E}$ determined by any rational section of $\mathcal{E}$). If $\mathcal{F}$ is a coherent subsheaf of $\mathcal{E}$, then $\mathcal{F}$ fits into an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

where $\mathcal{F}' = \mathcal{F} \cap \mathcal{E}'$ and $\mathcal{F}''$ is a subsheaf of $\mathcal{E}''$. We then have

$$\deg(\mathcal{F}) = \deg(\mathcal{F}') + \deg(\mathcal{F}'') \leq N(\mathcal{E}') + N(\mathcal{E}''),$$

so setting $N(\mathcal{E}) = N(\mathcal{E}') + N(\mathcal{E}'')$ satisfies the requirements of Lemma 13. □

Proof of Theorem 12. Let $\mathcal{E}$ be a vector bundle on X; we wish to show that $\mathcal{E}$ admits a Harder-Narasimhan filtration. We proceed by induction on the rank $\mathrm{rank}(\mathcal{E})$. Let S be the collection of all rational numbers of the form $\mathrm{slope}(\mathcal{E}')$, where $\mathcal{E}' \subseteq \mathcal{E}$ is a nonzero subbundle. It follows from Lemma 13 that S has a largest element. Let λ denote the largest element of S. Then there exists a nonzero subbundle $\mathcal{E}' \subseteq \mathcal{E}$ of slope λ. Choose such a subbundle whose rank is as large as possible. Note that $\mathcal{E}'$ is semistable of slope λ: it cannot admit a subbundle of larger slope, because that would contradict the maximality of λ.

Set $\mathcal{E}'' = \mathcal{E} / \mathcal{E}'$. Then $\mathcal{E}''$ is a vector bundle whose rank is smaller than $\mathcal{E}$. It follows from our inductive hypothesis that $\mathcal{E}''$ admits a Harder-Narasimhan filtration

$$0 = \mathcal{E}''_0 \subsetneq \mathcal{E}''_1 \subseteq \cdots \subsetneq \mathcal{E}''_m = \mathcal{E}'',$$

so that the slopes $\lambda_i = \mathrm{slope}(\mathcal{E}''_i / \mathcal{E}''_{i-1})$ form a decreasing sequence $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_m$. For $0 \leq i \leq m$, let $\overline{\mathcal{E}}''_i \subseteq \mathcal{E}$ denote the inverse image of $\mathcal{E}''_i$, so that $\overline{\mathcal{E}}''_0 = \mathcal{E}'$. We will complete the proof by showing that

$$0 \subsetneq \mathcal{E}' = \overline{\mathcal{E}}_0 \subsetneq \overline{\mathcal{E}}''_1 \subsetneq \cdots \subsetneq \overline{\mathcal{E}}''_m = \mathcal{E}$$

is a Harder-Narasimhan filtration of $\mathcal{E}$. By construction, the successive quotients of this filtration are given by $\mathcal{E}'$ and $\mathcal{E}''_i / \mathcal{E}''_i$, which are semistable of slopes λ and λ_i, respectively. It will therefore suffice to show that we have inequalities $\lambda > \lambda_1 > \lambda_2 > \cdots > \lambda_m$. Assume, for a contradiction, that this fails: that is, we have $\lambda \leq \lambda_1$. We have an exact sequence

$$0 \to \mathcal{E}' \to \overline{\mathcal{E}}''_1 \to \mathcal{E}''_1 \to 0,$$

satisfying $\mathrm{slope}(\mathcal{E}') = \lambda$ and $\mathrm{slope}(\mathcal{E}''_1) = \lambda_1$. Applying Exercise 1, we deduce that $\mathrm{slope}(\overline{\mathcal{E}}''_1) \geq \lambda$. This is impossible: we cannot have $\mathrm{slope}(\overline{\mathcal{E}}''_1) > \lambda$ (since λ was chosen to be the largest element of S), and we cannot have $\mathrm{slope}(\overline{\mathcal{E}}''_1) = \lambda$ (since $\mathcal{E}'$ was chosen to be maximal among subbundles of $\mathcal{E}$ having slope λ). □