Lecture 2: Tilting

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Let p be a prime number, which we regard as fixed throughout this lecture. In Lecture 1, we defined the *tilt* $K^\flat$ of an algebraically closed completely valued field K of residue characteristic p. In this lecture, we review the tilting construction in more detail, working in the more general setting of *perfectoid fields*.

Definition 1. A *perfectoid field* is a field K equipped with a nonarchimedean absolute value $| |_K : K \to \mathbf{R}_{\geq 0}$ satisfying the following axioms:

$(A1)$ The residue field $k = \mathcal{O}_K / \mathfrak{m}_K$ has characteristic p. Equivalently, the prime number p belongs to the maximal ideal $\mathfrak{m}_K$, so that $|p|_K < 1$.

$(A2)$ The field K is complete with respect to the absolute value $| |_K$.

$(A3)$ The Frobenius map $\varphi : \mathcal{O}_K / p\, \mathcal{O}_K \to \mathcal{O}_K / p\, \mathcal{O}_K$ is surjective. That is, for every element $x \in \mathcal{O}_K$, we can write $x = y^p + pz$ for some $y, z \in \mathcal{O}_K$.

$(A4)$ The maximal ideal $\mathfrak{m}_K$ is not generated by p. In other words, there exists some element $x \in K$ satisfying $|p|_K < |x|_K < 1$.

Remark 2. In the situation of Definition 1, choose $x \in K$ satisfying $|p|_K < |x|_K < 1$. Then $x \in \mathcal{O}_K$, so we can write $x = y^p + pz$ for some $y, z \in \mathcal{O}_K$. Since $|pz|_K \leq |p|_K < |x|_K$, we must have $|x|_K = |y^p|_K = |y|_K^p$. In particular, we have $|x|_K < |y|_K < 1$, so that $y \in \mathfrak{m}_K \setminus x\,\mathcal{O}_K$. It follows that the maximal ideal $\mathfrak{m}_K$ is not principal: that is, the valuation ring $\mathcal{O}_K$ is not a discrete valuation ring.

Remark 3. In the situation of Definition 1, suppose that K is characteristic p. In this case, axiom $(A1)$ is automatic, axiom $(A3)$ says that the field K is *perfect* (that is, every element of K has a pth root), and axiom $(A4)$ says that the absolute value $| |_K$ is nontrivial. In other words, a perfectoid field of characteirstic p is just a completely valued perfect field of characteristic p.

Example 4. Let K be a completely valued field of residue characteristic p. Suppose that every element $x \in K$ has a pth root (this condition is satisfied, for example, if K is algebraically closed). Then axioms $(A3)$ and $(A4)$ are satisfied, so K is a perfectoid field.

Example 5. For each $n > 0$, let $\mathbf{Z}[\zeta_{p^n}]$ denote ring obtained from $\mathbf{Z}$ by adjoining a primitive p^nth root of unity, given by the quotient $\mathbf{Z}[x]/(1 + x^{p^{n-1}} + x^{2p^{n-1}} + \cdots + x^{(p-1)p^{n-1}})$; equivalently $\mathbf{Z}[\zeta_{p^n}]$ can be described as the ring of integers in the number field $\mathbf{Q}(\zeta_{p^n})$.

Let $\mathbf{Z}_p^{\mathrm{cyc}}$ denote the p-adic completion of the union $\bigcup_{n>0} \mathbf{Z}[\zeta_{p^n}]$ and set $\mathbf{Q}_p^{\mathrm{cyc}} = \mathbf{Z}_p^{\mathrm{cyc}}[1/p]$. Then $K = \mathbf{Q}_p^{\mathrm{cyc}}$ is a perfectoid field with ring of integers $\mathcal{O}_K = \mathbf{Z}_p^{\mathrm{cyc}}$. Axiom $(A3)$ follows from the observation that the image of the Frobenius map

$$\varphi : \mathbf{Z}_p^{\mathrm{cyc}}/p\mathbf{Z}_p^{\mathrm{cyc}} \to \mathbf{Z}_p^{\mathrm{cyc}}/p\mathbf{Z}_p^{\mathrm{cyc}}$$

is a subgroup of $\mathbf{Z}_p^{\mathrm{cyc}}/p\mathbf{Z}_p^{\mathrm{cyc}} \simeq \bigcup_{n>0} \mathbf{F}_p[\zeta_{p^n}]$ which contains each of the roots of unity ζ_{p^n}, by virtue of the equation $\zeta_{p^n} = (\zeta_{p^{n+1}})^p$.

Note that the pth power map $\mathbf{Q}_p^{\mathrm{cyc}} \to \mathbf{Q}_p^{\mathrm{cyc}}$ is *not* surjective: for example, there is no element $x \in \mathbf{Q}_p^{\mathrm{cyc}}$ satisfying $x^p = p$.

As in the previous lecture, we let $K^\flat$ denote the inverse limit of the system

$$\cdots \to K \xrightarrow{x \mapsto x^p} K \xrightarrow{x \mapsto x^p} K,$$

whose elements can be identified with sequences $\vec{x} = \{x_0, x_1, \ldots \in K : x_n = x_{n+1}^p\}$. We regard $K^\flat$ as a monoid with respect to the obvious multiplication

$$\{x_n\}_{n \geq 0} \cdot \{y_n\}_{n \geq 0} = \{x_n \cdot y_n\}_{n \geq 0}.$$

When K is a perfectoid field, we can equip $K^\flat$ with a compatible addition law. To prove this, it is convenient to first work with the subset $\mathcal{O}_K^\flat \subseteq K^\flat$ consisting of those sequences $\{x_n\}_{n \geq 0}$ where each x_n belongs to $\mathcal{O}_K$ (note that if this condition is satisfied for any integer $n \geq 0$, then it is satisfied for all integers $n \geq 0$).

Proposition 6. *Let K be a completely valued field of residue characteristic p. Then canonical map $\mathcal{O}_K \to \mathcal{O}_K / p\, \mathcal{O}_K$ induces a bijection*

$$\mathcal{O}_K^\flat \to \varprojlim (\cdots \to \mathcal{O}_K / p\, \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K / p\, \mathcal{O}_K)$$

Proof. Let us assume that K has characteristic zero (in characteristic p, there is nothing to prove). Our assumption that K is complete implies that $\mathcal{O}_K$ can be realized as the inverse limit $\varprojlim_n \mathcal{O}_K / p^n\, \mathcal{O}_K$. For each $n \geq 1$, let $Z(n)$ denote the limit of the inverse system of sets

$$\cdots \to \mathcal{O}_K / p^n\, \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K / p^n\, \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K / p^n\, \mathcal{O}_K \xrightarrow{x \mapsto x^p} \mathcal{O}_K / p^n\, \mathcal{O}_K .$$

Then $\mathcal{O}_K^\flat$ is the inverse limit $\varprojlim_n Z(n)$, and we wish to show that the projection map $\mathcal{O}_K^\flat \to Z(1)$ is a bijection. For this, it will suffice to show that each of the transition maps $Z(n) \to Z(n-1)$ is a bijection. In other words, it will suffice to show that the vertical maps in the diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathcal{O}_K / p^n\, \mathcal{O}_K & \xrightarrow{(\bullet)^p} & \mathcal{O}_K / p^n\, \mathcal{O}_K & \xrightarrow{(\bullet)^p} & \mathcal{O}_K / p^n\, \mathcal{O}_K \\
 & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
\cdots \longrightarrow & \mathcal{O}_K / p^{n-1}\, \mathcal{O}_K & \xrightarrow{(\bullet)^p} & \mathcal{O}_K / p^{n-1}\, \mathcal{O}_K & \xrightarrow{(\bullet)^p} & \mathcal{O}_K / p^{n-1}\, \mathcal{O}_K
\end{array}$$

induce an isomorphism after taking the inverse limit in the horizontal direction. For this, we note the existence (and uniqueness) of dotted arrows rendering the diagram commutative: this comes from the elementary observation that for $x, y \in \mathcal{O}_K$, we have

$$(x \equiv y \pmod{p^{n-1}}) \Rightarrow (x^p \equiv y^p \pmod{p^n}).$$

□

Corollary 7. *Let K be a completely valued field of residue characteristic p. Then we can equip $\mathcal{O}_K^\flat$ with the structure of a commutative ring, where the multiplication is defined pointwise and the addition is uniquely determined by the requirement that*

$$\{x_n\}_{n \geq 0} + \{y_n\}_{n \geq 0} = \{z_n\}_{n \geq 0} \Rightarrow x_n + y_n \equiv z_n \pmod{p}.$$

Remark 8. In the situation of Corollary 7, we can describe the addition law on $\mathcal{O}_K^\flat$ more explicitly. Suppose we are given elements $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ in $\mathcal{O}_K^\flat$. Write $\{x_n\}_{n \geq 0} + \{y_n\}_{n \geq 0} = \{z_n\}_{n \geq 0}$, so that we have $x_m + y_m \equiv z_m \pmod{p}$ for each $n \geq 0$. Writing $z_m = x_m + y_m + pw$ for some $w \in \mathcal{O}_K$, we obtain

$$\begin{aligned}
z_0 &= z_m^{p^m} \\
&= (x_m + y_m + pw)^{p^m} \\
&= \sum_{i=0}^{p^m} \binom{p^m}{i} (pw)^i (x_m + y_m)^{p^m - i} \\
&\equiv (x_m + y_m)^{p^m} \pmod{p^m}.
\end{aligned}$$

It follows that z_0 is given concretely as the limit $\lim_{m\to\infty}(x_m + y_m)^{p^m}$. More generally, each z_n is given concretely as $\lim_{m\to\infty}(x_{n+m} + y_{n+m})^{p^m}$.

Note that, to prove Proposition 6, we do not need to assume that K is a perfectoid field: it is enough to assume axioms $(A1)$ and $(A2)$ of Definition 1. However, at this level of generality, the tilt $K^\flat$ might be "too small."

Exercise 9. Let $K = \mathbf{Q}_p$ be the field of p-adic rational numbers, equipped with the usual p-adic absolute value. Show that $K^\flat = \mathcal{O}_K^\flat$ is isomorphic to $\mathbf{F}_p$.

Our next goal is to show that, when K is a perfectoid field, the tilt $K^\flat$ is very large (Proposition 13).

Notation 10. Let K be a completely valued field of residue characteristic p and let $x = \{x_n\}_{n\geq 0}$ be an element of $K^\flat$. We set $x^\sharp = x_0 \in K$. The construction $x \mapsto x^\sharp$ then determines a multiplicative map $\sharp : K^\flat \to K$. For each $x \in K^\flat$, we define $|x|_{K^\flat} = |x^\sharp|_K$.

Example 11. Suppose that K is algebraically closed (or, more generally, that every element of K admits a pth root). Then the map $x \mapsto x^\sharp$ determines a surjection $K^\flat \to K$,

Example 12. Suppose that K is a perfect field of characteristic p. Then the map $\sharp : K^\flat \to K$ is bijective.

Proposition 13. *Let K be a perfectoid field. Then:*

(1) *For every element $x \in \mathcal{O}_K$, there exists an element $x' \in \mathcal{O}_K^\flat$ satisfying $x \equiv x'^\sharp \pmod p$.*

(2) *For every element $y \in K$, there exists an element $y' \in K^\flat$ satisfying $|y|_K = |y'|_{K^\flat}$.*

Proof. Assertion (1) follows from Proposition 6 together with the observation that, if K satisfies axiom $(A3)$, then the transition maps in the diagram

$$\cdots \to \mathcal{O}_K / p\,\mathcal{O}_K \xrightarrow{x\mapsto x^p} \mathcal{O}_K / p\,\mathcal{O}_K \xrightarrow{x\mapsto x^p} \mathcal{O}_K / p\,\mathcal{O}_K$$

are surjective.

To prove (2), we may assume without loss of generality we may assume that $y \neq 0$. Using axiom $(A4)$ of Definition 1, we can choose an element $x \in K$ with $|p|_K < |x|_K < 1$. Replacing x by an element which is congruent modulo p, we can assume that $x = x'^\sharp$ for some $x' \in K^\flat$ (by virtue of (1)). We are therefore free to modify y by multiplying it by a suitable power of x, and can therefore reduce to the case where $|x|_K \leq |y|_K < 1$. In this case, we have $|p|_K < |y|_K < 1$. Using part (1) again, we can choose $y' \in K^\flat$ with $y'^\sharp \equiv y \pmod p$, so that $|y|_K = |y'^\sharp|_K = |y'|_{K^\flat}$. □

Exercise 14. Show that the converse of Proposition 13 is also true: if K is a completely valued field of residue characteristic p, then assertion (1) of Proposition 13 implies that K satisfies axiom $(A3)$ of Definition 1, and assertion (2) of Proposition 13 implies that K satisfies axiom $(A4)$ of Definition 1. In other words, the axioms for a perfectoid field are exactly what we need to guarantee that the tilt $K^\flat$ is "sufficiently large."

Using Proposition 13, we can choose an element π in $K^\flat$ such that $0 < |\pi|_{K^\flat} < 1$. For each $n \in \mathbf{Z}$, we have

$$\pi^{-n}\,\mathcal{O}_K^\flat = \{x \in K^\flat : |x|_{K^\flat} \leq |\pi|_{K^\flat}^{-n}\}$$

It follows that, as a set, we can identify $K^\flat$ with the direct limit

$$\mathcal{O}_K^\flat \xrightarrow{\pi} \mathcal{O}_K^\flat \xrightarrow{\pi} \mathcal{O}_K^\flat \xrightarrow{\cdots},$$

where the transition maps are given by multiplication by π. This proves the following:

Proposition 15. *Let K be a perfectoid field. Then the inclusion $\mathcal{O}_K^\flat \hookrightarrow K^\flat$ extends uniquely to a multiplicative bijection $\mathcal{O}_K^\flat[\pi^{-1}] \simeq K^\flat$. Consequently, there is a unique ring structure on $K^\flat$ which is compatible with its multiplication and which coincides, on $\mathcal{O}_K^\flat$, with the ring structure of Corollary 7.*

Exercise 16. Show that the addition law on $K^\flat$ is given in general by the formula

$$\{x_n\}_{n\geq 0} + \{y_n\}_{n\geq 0} = \{\lim_{m\to\infty}(x_{m+n} + y_{m+n})^{p^m}\}_{n\geq 0}$$

Theorem 17. *Let K be a perfectoid field. Then $K^\flat$, with the ring structure of Proposition 15 and the map $|\,|_{K^\flat} : K^\flat \to \mathbf{R}_{\geq 0}$, is a perfectoid field of characteristic p.*

Proof. Note that if $\{x_n\}_{n\geq 0}$ is nonzero element of $K^\flat$, then each x_n is a nonzero element of K; it follows that $\{x_n^{-1}\}_{n\geq 0}$ is also an element of $K^\flat$ which is a multiplicative inverse for $\{x_n\}_{n\geq 0}$. This proves that $K^\flat$ is a field. Proposition 6 realizes $\mathcal{O}_K^\flat$ as an inverse limit of copies of $\mathcal{O}_K/p\,\mathcal{O}_K$ (with transition maps given by the Frobenius). Since p vanishes in $\mathcal{O}_K/p\,\mathcal{O}_K$, it vanishes in $\mathcal{O}_K^\flat$ and therefore also in $K^\flat$: that is, $K^\flat$ is a field of characteristic p. We claim that $|\,|_{K^\flat}$ is a non-archimedean absolute value on $K^\flat$. The identities

$$|0|_{K^\flat} = 0 \qquad |1|_{K^\flat} = 1 \qquad |x\cdot y|_{K^\flat} = |x|_{K^\flat}\cdot|y|_{K^\flat}$$

are immediate from the definition. It will therefore suffice to show that for $x = \{x_n\}_{n\geq 0}$ and $y = \{y_n\}_{n\geq 0} \in K^\flat$, we have

$$|x+y|_{K^\flat} \leq \max(|x|_{K^\flat}, |y|_{K^\flat}).$$

Using the formula of Exercise 16, we are reduced to proving that

$$|(x_m + y_m)^{p^m}|_K \leq \max(|x_m|_K^{p^m}, |y_m|_K^{p^m}),$$

which follows (after extracting p^mth roots) from the analogous fact for the absolute value $|\,|_K$.

The field $K^\flat$ is perfect by construction: every element $(x_0, x_1, x_2, \ldots) \in K^\flat$ has a unique pth root, given by the shifted sequence $(x_1, x_2, x_3, \ldots) \in K^\flat$. Moreover, the absolute value on $K^\flat$ is nontrivial because it takes the same values as the absolute value on K (Proposition 13). We will complete the proof by showing that $K^\flat$ is complete. Let us assume that K has characteristic zero (if K has characteristic p, then the map $\sharp : K^\flat \to K$ is an isomorphism of valued fields and there is nothing to prove). Using Proposition 13, we can choose an element $\pi \in K^\flat$ satisfying $|\pi|_{K^\flat} = |p|_K$. We wish to show that the ring $\mathcal{O}_K^\flat$ is π-adically complete: that is, that it can be realized as the inverse limit of the system

$$\cdots \to \mathcal{O}_K^\flat/(\pi^{p^3}) \to \mathcal{O}_K^\flat/(\pi^{p^2}) \to \mathcal{O}_K^\flat/(\pi^p) \to \mathcal{O}_K^\flat/(\pi).$$

For each $m \geq 0$, the map of sets

$$\mathcal{O}_K^\flat \to \mathcal{O}_K \qquad (x = \{x_n\}_{n\geq 0}) \mapsto (x_m = (x^{1/p^m})^\sharp)$$

induces a ring homomorphism $\mathcal{O}_K^\flat \to \mathcal{O}_K/p\,\mathcal{O}_K$ which annihilates π^{p^m}, and therefore factors through a map $u_m : \mathcal{O}_K^\flat/(\pi^{p^m}) \to \mathcal{O}_K/p\,\mathcal{O}_K$. These maps fit into a commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathcal{O}_K^\flat/(\pi^{p^2}) & \longrightarrow & \mathcal{O}_K^\flat/(\pi^p) & \longrightarrow & \mathcal{O}_K^\flat/(\pi) \\
& \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} \\
\cdots \longrightarrow & \mathcal{O}_K/p\,\mathcal{O}_K & \xrightarrow{\varphi} & \mathcal{O}_K/p\,\mathcal{O}_K & \xrightarrow{\varphi} & \mathcal{O}_K/p\,\mathcal{O}_K
\end{array}$$

where the inverse limit of the lower diagram agrees with $\mathcal{O}_K^\flat$ by virtue of Proposition 6. It will therefore suffice to show that each of the maps u_m is an isomorphism. This reduces immediately to the case $m = 0$, where it is a special case of Lemma 18 below. □

Lemma 18. *Let K be a perfectoid field and let $\pi \in K^\flat$ be a nonzero element satisfying $|p|_K \leq |\pi|_{K^\flat} < 1$. Then the map $\sharp : K^\flat \to K$ induces an isomorphism $\mathcal{O}_K^\flat/(\pi) \to \mathcal{O}_K/(\pi^\sharp)$.*

Proof. Surjectivity follows from Proposition 13. To prove injectivity, we note that if $x \in \mathcal{O}_K^\flat$ has the property that $x^\sharp \equiv 0 \pmod{\pi^\sharp}$, then $|x|_{K^\flat} = |x^\sharp|_K \leq |\pi^\sharp|_K = |\pi|_{K^\flat}$ so that x is divisibly by π in $\mathcal{O}_K^\flat$. □