Lecture 19: Line Bundles on the Fargues-Fontaine Curve and Their Cohomology

November 18, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let X denote the Fargues-Fontaine curve, given by

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} B^{\varphi = p^n}).$$

We have seen that X is a Dedekind scheme. Our first goal in this lecture is to describe the Picard group $\mathrm{Pic}(X)$ consisting of isomorphism classes of line bundles on X.

Construction 1. By general nonsense, every graded module M over the graded ring $\bigoplus_{n \geq 0} B^{\varphi = p^n}$ determines a quasi-coherent sheaf $\widetilde{M}$ on X. If $x \in X$ is a closed point given by the vanishing of a function $\log([\epsilon]) \in B^{\varphi = p}$ and $U = X - \{x\}$ is the complementary affine open set, then $\widetilde{M}(U)$ is the degree zero part of the graded module $M[\frac{1}{\log([\epsilon])}]$.

In particular, we can apply this construction to the graded module $M = \bigoplus_{n \in \mathbf{Z}} B^{\varphi = p^{n+1}}$ (obtained from $\bigoplus_{n \in \mathbf{Z}} B^{\varphi = p^n}$ by shifting the grading). This yields a line bundle $\mathcal{O}(1)$ on X, given on the affine subset above by the formula

$$\mathcal{O}(1)(U) = M[\frac{1}{\log([\epsilon])}]^0 = (B[\frac{1}{\log([\epsilon])}]^{\varphi = p}).$$

More generally, for any integer m, we can consider the line bundle $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$ on X, given by

$$\mathcal{O}(m)(U) = (B[\frac{1}{\log([\epsilon])}]^{\varphi = p^m}).$$

Remark 2. If M is any graded module over the ring $\bigoplus_{n \geq 0} B^{\varphi = p^n}$ and M^0 is the degree zero part of M, then we have a canonical map

$$M^0 \to \mathrm{H}^0(X; \widetilde{M}).$$

In particular, the line bundles $\mathcal{O}(m)$ are equipped with canonical maps

$$\mu : B^{\varphi = p^m} \to \mathrm{H}^0(X; \mathcal{O}(m)).$$

We now prove the following:

Theorem 3. *The construction $m \mapsto \mathcal{O}(m)$ induces an isomorphism of abelian groups $\rho : \mathbf{Z} \to \mathrm{Pic}(X)$.*

Let $\mathrm{Div}(X)$ denote the group of divisors on X: that is, the free abelian group generated by the set of closed points of X. There is a canonical map

$$\mathrm{Div}(X) \to \mathrm{Pic}(X),$$

which carries a closed point $x \in X$ to the line bundle $\mathcal{O}(x)$ given by the *inverse* of the ideal sheaf of X (which is an invertible sheaf, since X is a Dedekind scheme). On the other hand, we also have a homomorphism $\deg : \mathrm{Div}(X) \to \mathbf{Z}$, given by $\deg(x) = 1$ for each closed point $x \in X$.

Lemma 4. *The diagram*

$$\begin{array}{ccc} & \mathrm{Div}(X) & \\ {}^{\deg}\swarrow & & \searrow \\ \mathbf{Z} & \xrightarrow{\rho} & \mathrm{Pic}(X) \end{array}$$

is commutative.

Proof. It will suffice to show that, for every closed point $x \in X$, the line bundle $\mathcal{O}(x)$ is isomorphic to the line bundle of $\mathcal{O}(1)$ of Construction 1. Choose $\epsilon \in 1 + \mathfrak{m}_C^\flat$ such that $\log([\epsilon])$ vanishes at x. Under the canonical map

$$B^{\varphi=p} \to \mathrm{H}^0(X; \mathcal{O}(1)),$$

we can view $\log([\epsilon])$ as a global section of $\mathcal{O}(1)$ which vanishes to order 1 at the point x, and therefore extends to an isomorphism $\mathcal{O}(x) \simeq \mathcal{O}(1)$. □

For any Dedekind scheme X, the canonical map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is surjective. It follows from Lemma 4 that the map $\rho : \mathbf{Z} \to \mathrm{Pic}(X)$ is also surjective. To complete the proof of Theorem 3, it will suffice to show that ρ is injective: that is, that the line bundles $\mathcal{O}(m)$ are nontrivial whenever m is nonzero. This is a consequence of the following more precise assertion, which describes the cohomology of the line bundles $\mathcal{O}(m)$:

Theorem 5. (*a*) *For each integer m, the canonical map $\mu : B^{\varphi=p^m} \to \mathrm{H}^0(X; \mathcal{O}(m))$ is an isomorphism.*

(*b*) *For $m > 0$, the cohomology groups $\mathrm{H}^i(X; \mathcal{O}(m))$ vanish for $i > 0$.*

Corollary 6. *The canonical map $\mathbf{Q}_p \to \mathrm{H}^0(X, \mathcal{O}_X)$ is an isomorphism, and the cohomology group $\mathrm{H}^1(X, \mathcal{O}_X)$ vanishes.*

Remark 7. Since X is a Dedekind scheme, the cohomology groups $\mathrm{H}^i(X; \mathcal{O}(m))$ automatically vanish for $i > 1$. Beware, however, that the cohomology groups $\mathrm{H}^1(X; \mathcal{O}(m))$ do not vanish for $m < 0$.

Remark 8. Let Y be a (complete) algebraic curve over an algebraically closed field k. Then Y has genus zero if and only either of the equivalent conditions is satisfied;

- The Picard group $\mathrm{Pic}(Y)$ is isomorphic to $\mathbf{Z}$.
- The cohomology group $\mathrm{H}^1(Y; \mathcal{O}_Y)$ vanishes.

Theorems 3 and 5 assert that these properties hold for the Fargues-Fontaine curve X. This supports the heuristic that X is "like" an algebraic curve of genus 0.

Proof of Theorem 5. Let t be a nonzero element of $B^{\varphi=p}$ (so that we can write $t = \log([\epsilon])$ for some unique $\epsilon \in 1 + \mathfrak{m}_C^\flat$), so that t determines a global section of $\mathcal{O}(1)$ (which we will also denote by t). This section vanishes at a single point $x \in X$. For any integer m, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^{\varphi=p^m} & \xrightarrow{t} & B^{\varphi=p^{m+1}} & \longrightarrow & B^{\varphi=p^{m+1}}/tB^{\varphi=p^m} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & \mathrm{H}^0(X, \mathcal{O}(m)) & \longrightarrow & \mathrm{H}^0(X, \mathcal{O}(m+1)) & \longrightarrow & \mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m)). & & \end{array}$$

We first prove the following:

($*$) If $m \geq 0$, then the map $\nu : B^{\varphi=p^{m+1}}/tB^{\varphi=p^m} \to \mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m))$ is an isomorphism.

Recall that every nonzero element of $B^{\varphi=p^{m+1}}$ factors as a product $t_0 t_1 \cdots t_m$, where each t_i is a nonzero element of $B^{\varphi=p}$ vanishing at a single point $x_i \in X$. Such an element is annihilated by the composite map

$$B^{\varphi=p^{m+1}} \to B^{\varphi=p^{m+1}}/tB^{\varphi=p^m} \xrightarrow{\nu} \mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m))$$

if and only if the product $t_0 t_1 \cdots t_m$ vanishes at the point x: that is, if and only if $x = x_i$ for some i. In this case, t_i is a unit multiple of t, so that $t_0 \cdots t_m$ belongs to $tB^{\varphi=p^m}$. This proves that ν is injective.

To verify the surjectivity of ν, we can use the commutativity of the diagram

$$\begin{array}{ccc}
B^{\varphi=p}/tB^{\varphi=1} & \xrightarrow{t^m} & B^{\varphi=p^{m+1}}/tB^{\varphi=p^m} \\
\downarrow \nu & & \downarrow \nu \\
\mathrm{H}^0(X, \mathcal{O}(1)/t\,\mathcal{O}(0)) & \xrightarrow[\sim]{t^m} & \mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m))
\end{array}$$

to reduce to the case $m = 0$. In this case, the assertion is equivalent to the surjectivity of the map

$$B^{\varphi=p} \simeq 1 + \mathfrak{m}_C^\flat \xrightarrow{\epsilon \mapsto \log(\epsilon^\sharp)} K_x$$

(where K_x is the untilt of $C^\flat$ corresponding to the point x), which was established in Lecture 10.

Since $\mathcal{O}(m+1)/t\,\mathcal{O}(m)$ is supported at a single point of X, its cohomology vanishes in degree 1. We therefore have a long exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{O}(m)) \to \mathrm{H}^0(X, \mathcal{O}(m+1)) \to u\mathrm{H}^0(X, \mathcal{O}(m+1)/t\,\mathcal{O}(m)) \to \mathrm{H}^1(X; \mathcal{O}(m)) \to \mathrm{H}^1(X; \mathcal{O}(m+1)) \to 0$$

It follows from ($*$) that the map u is surjective for $m \geq 0$. Combining this observation with ($*$) and the snake lemma, we deduce (again for $m \geq 0$) that multiplication by t induces isomorphisms

$$\ker(B^{\varphi=p^m} \to \mathrm{H}^0(X, \mathcal{O}(m))) \to \ker(B^{\varphi=p^{m+1}} \to \mathrm{H}^0(X, \mathcal{O}(m+1)))$$

$$\mathrm{coker}(B^{\varphi=p^m} \to \mathrm{H}^0(X, \mathcal{O}(m))) \to \mathrm{coker}(B^{\varphi=p^{m+1}} \to \mathrm{H}^0(X, \mathcal{O}(m+1)))$$
$$\mathrm{H}^1(X, \mathcal{O}(m)) \to \mathrm{H}^1(X, \mathcal{O}(m+1)).$$

Passing to the limit, we conclude that the kernel and cokernel of μ coincide with the kernel and cokernel of the canonical isomorphism

$$(B[\frac{1}{t}])^{\varphi=1} \to \varinjlim_{n \geq m} \mathrm{H}^0(X, \mathcal{O}(nx)) = \mathrm{H}^0(U, \mathcal{O}_U),$$

where $U = X - \{x\}$ is the affine open subset of X complementary to the vanishing locus of t. Similarly, the canonical map

$$\mathrm{H}^1(X, \mathcal{O}(m)) \to \varinjlim_{n \geq m} \mathrm{H}^1(X, \mathcal{O}(nx)) \simeq \mathrm{H}^1(U, \mathcal{O}_U) \simeq 0$$

is an isomorphism, so $\mathrm{H}^1(X, \mathcal{O}(m))$ vanishes. This proves Theorem 5 in the case $m \geq 0$.

To handle the case $m < 0$, it will suffice to show that the cohomology groups $\mathrm{H}^0(X, \mathcal{O}(-n)) = \mathrm{H}^0(X, \mathcal{O}(-nx))$ vanish for $n > 0$. To prove this, we observe that a nonzero element of $\mathrm{H}^0(X, \mathcal{O}(-nx))$ can be identified with a nonzero element of $\mathrm{H}^0(X, \mathcal{O}_X)$ vanishing to order n at the point x. However, the first part of the proof shows that $\mathrm{H}^0(X, \mathcal{O}_X) \simeq B^{\varphi=1} \simeq \mathbf{Q}_p$ is a field, so a nonzero element of $\mathrm{H}^0(X, \mathcal{O}_X)$ cannot vanish at any point of X. $\square$

Definition 9. We let $\deg : \mathrm{Pic}(X) \to \mathbf{Z}$ denote the inverse of the isomorphism $\mathbf{Z} \to \mathrm{Pic}(X)$ appearing in Theorem 3. If $\mathcal{L}$ is a line bundle on X, we will refer to $\deg(\mathcal{L})$ as the *degree* of $\mathcal{L}$.

More generally, if $\mathcal{E}$ is a vector bundle of rank r on X, we let $\deg(\mathcal{E})$ denote the *degree* of $\mathcal{E}$, defined by the formula

$$\deg(\mathcal{E}) = \deg(\bigwedge^r \mathcal{E}).$$

Note that a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of vector bundles of ranks r', r, and r'' determines a canonical isomorphism

$$\bigwedge^r(\mathcal{E}) \simeq (\bigwedge^{r'} \mathcal{E}') \otimes (\bigwedge^{r''} \mathcal{E}''),$$

hence an equality $\deg(\mathcal{E}) = \deg(\mathcal{E}') + \deg(\mathcal{E}'')$.

For any vector bundle $\mathcal{E}$ on X, we define the *slope* of $\mathcal{E}$ by the formula

$$\mathrm{slope}(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$