Lecture 18: Bounded and Meromorphic Functions of p

November 18, 2018

Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p. Our goal in this lecture is to give an "intrinsic" description of $\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$ as a subring of B: roughly speaking, it consists of "holomorphic" functions on Y whose value at any point $y \in Y$ belongs to the valuation ring $\mathcal{O}_{K_y}$ of the perfectoid field K_y corresponding to y.

Theorem 1. *Let f be a nonzero element of B. The following conditions are equivalent:*

(1) *For each $\rho \in (0,1)$, we have $|f|_\rho \leq 1$.*

(2) *The element f belongs to the subring $\mathbf{A}_{\mathrm{inf}} \subseteq B$.*

Corollary 2. *Let f be a nonzero element of B. Then:*

- *The element f belongs to the localization $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}]$ if and only if there exists an integer n such that $|f|_\rho \leq \rho^n$ for all $\rho \in (0,1)$.*
- *The element f belongs to the localization $\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\pi]}]$ if and only if there exists a constant $C > 0$ satisfying $|f|_\rho \leq C$ for all $\rho \in (0,1)$.*
- *The element f belongs to the localization $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$ if and only if there exists a constant $C > 0$ and an integer n satisfying $|f|_\rho \leq C\rho^n$ for all $\rho \in (0,1)$.*

We will deduce Theorem 1 from the following weaker assertion:

Lemma 3. *Let f be an element of B. Suppose that there exists an integer m such that*

$$|f|_\rho \leq \rho^m$$

for all $0 < \rho < 1$. Then we can write

$$f = [c]p^m + g,$$

where $c \in \mathcal{O}_C^\flat$ and g satisfies an inequality of the form $|g|_\rho \leq \rho^{m+1}$.

Proof of Theorem 1 from Lemma 3. The implication $(2) \Rightarrow (1)$ is immediate. Conversely, suppose that (1) is satisfied, and set $f_0 = f$. Applying Lemma 3, we can write $f_0 = [c_0] + f_1$, where $[c_0] \in \mathcal{O}_C^\flat$, and f_1 satisfies $|f_1|_\rho \leq \rho$ for all $\rho \in (0,1)$. Applying Lemma 3 again, we can write $f_1 = [c_1]p + f_2$, where $c_1 \in \mathcal{O}_C^\flat$ and f_2 satisfies $|f_2|_\rho \leq \rho$ for all $\rho \in (0,1)$. Continuing in this way, we obtain a sequence of elements $f_0, f_1, f_2, \ldots \in B$ and $c_0, c_1, c_2, \ldots \in \mathcal{O}_C^\flat$ satisfying

$$f_0 = [c_0] + [c_1]p + \cdots + [c_{n-1}]p^{n-1} + f_n$$

$$|f_n|_\rho \leq \rho^n.$$

Note that the sequence $\{f_n\}_{n \geq 0}$ converges to zero with respect to the each of the Gauss norms $| \bullet |_\rho$. It follows that the infinite sum $\sum_{n \geq 0} [c_n]p^n$ converges in B to f, so that f belongs to $\mathbf{A}_{\mathrm{inf}}$ as desired. $\square$

Proof of Lemma 3. Replacing f by $\frac{f}{p^m}$, we can reduce to the case $m = 0$. In this case, we have an element $f \in B$ satisfying $|f|_\rho \leq 1$ for all $\rho \in (0, 1)$; we wish to write $f = [c] + g$ for $c \in \mathcal{O}_C^\flat$, where g satisfies $|g|_\rho \leq \rho$ for all $\rho \in (0, 1)$.

Choose a sequence $f_1, f_2, \ldots \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ which converges to f in B. Each f_i then admits a unique Teichmüller expansion

$$f_i = \sum_{n \gg -\infty} [c_{n,i}] p^n.$$

Set $f_i^+ = \sum_{n \geq 0} [c_{n,i}] p^n$. We claim that the sequence $f_1^+, f_2^+, \ldots$ also converges to f in B. To prove this, we must show that for each $\rho \in (0, 1)$, we have

$$\lim_{i \to \infty} |f_i - f_i^+|_\rho = 0.$$

Let ϵ be a small positive real number. Then the sequence $f_1, f_2, \ldots$ converges to f with respect to the Gauss norm $| \bullet |_{\epsilon \cdot \rho}$. It follows that, for i sufficiently large (depending on ϵ), we have

$$|f_i|_{\epsilon \cdot \rho} = |f|_{\epsilon \cdot \rho} \leq 1.$$

For such i, we have

$$|c_{-n,i}|_{C^\flat} (\epsilon \rho)^{-n} \leq 1.$$

If n is positive, this gives

$$|c_{-n,i}|_{C^\flat} \rho^{-n} \leq \epsilon^n \leq \epsilon.$$

We therefore have

$$|f_i - f_i^+|_\rho = \sup_{n > 0} (|c_{-n,i}|_{C^\flat} \rho^{-n}) \leq \epsilon$$

for sufficiently large i.

Replacing the sequence $\{f_i\}$ with $\{f_i^+\}$, we may assume that each f_i admits a Teichmüller expansion of the form

$$f_i = \sum_{n \geq 0} [c_{n,i}] p^n.$$

Then, for every pair of indices i and j, the difference $f_i - f_j$ admits a Teichmüller expansion of the form $[c_{0,i} - c_{0,j}] +$ higher order terms. For any $\rho \in (0, 1)$, we have

$$|f_i - f_j|_\rho \geq |c_{0,i} - c_{0,j}|_{C^\flat}.$$

Since the sequence $\{f_i\}$ is Cauchy with respect to the Gauss norm $| \bullet |_\rho$, it follows that $\{c_{0,i}\}$ is a Cauchy sequence in the field $C^\flat$. Since $C^\flat$ is complete, this Cauchy sequence converges to some element $c \in C^\flat$. Moreover, for $i \gg 0$, we have

$$|c_{0,i}|_{C^\flat} \leq |f_i|_\rho = |f|_\rho \leq 1,$$

so that $c_{0,i}$ belongs to $\mathcal{O}_C^\flat$ (for $i \gg 0$) and therefore $c \in \mathcal{O}_C^\flat$.

Exercise 4. Show that, if $\{c_i\}$ is a Cauchy sequence in $\mathcal{O}_C^\flat$ converging to a point $c \in \mathcal{O}_C^\flat$, then we have $[c] = \lim_{i \to \infty} [c_i]$ in the ring B.

For each i, set $g_i = f_i - [c_{0,i}] = \sum_{n > 0} [c_{n,i}] p^i$. Applying the exercise, we see that the limit $\lim_{i \to \infty} g_i$ exists and is given by

$$\lim_{i \to \infty} g_i = (\lim_{i \to \infty} f_i) - (\lim_{i \to \infty} [c_{0,i}]) = f - [c].$$

That is, we can write $f = [c] + g$, where $g = \lim_{i \to \infty} g_i$. We will complete the proof by showing that $|g|_\rho \leq \rho$ for all $\rho \in (0, 1)$, or equivalently that $v_s(g) \geq s$ for all $s \in \mathbf{R}_{>0}$.

Let us assume that $g \neq 0$ (otherwise there is nothing to prove). Passing to a subsequence, we may then also assume that $g_i \neq 0$ for all i. Each g_i admits a Teichmüller expansion where only positive powers of p occur, so that the piecewise linear function $v_\bullet(g_i)$ has strictly positive slopes. When restricted to any compact interval $I \subseteq \mathbf{R}_{>0}$, the function $v_\bullet(g)$ agrees with $v_\bullet(g_i)$ for $i \gg 0$. It follows that the piecewise linear function $s \mapsto v_s(g)$ also has strictly positive (and integral) slopes. Suppose, for a contradiction, that there exists some $s > 0$ such that $v_s(g) < s$. Choose $0 < s' < s$ such that $v_s(g) - s + s' < 0$. Since the function $v_\bullet(g)$ is piecewise linear with slopes ≥ 1 everywhere, we have

$$v_{s'}(g) \leq v_s(g) - s + s' < 0.$$

Setting $\rho' = e^{-s'}$, we have $|g|_{\rho'} > 1$. Then

$$1 < |g|_{\rho'} = |f - [c]|_{\rho'} \leq \max(|f|_{\rho'}, |[c]|_{\rho'}) = \max(|f|_{\rho'}, |c|_{C^\flat}) \leq 1,$$

which is a contradiction. $\square$

From Theorem 1, it is easy to describe the invariant subring $B^{\varphi=1} \subseteq B$:

Theorem 5. *The unit map $\mathbf{Q}_p \to B^{\varphi=1}$ is an isomorphism.*

Lemma 6. *Let f be a nonzero element of $B^{\varphi=1}$. Then there exists an integer n such that $|f|_\rho = \rho^n$ for all $0 < \rho < 1$.*

Proof. Note that for $0 < \rho < 1$, we have

$$|f|_\rho^p = |\varphi(f)|_{\rho^p} = |f|_{\rho^p}.$$

In other words, the function $s \mapsto v_s(f)$ satisfies the identity $v_{ps}(f) = pv_s(f)$. Differentiating both sides (on the left) with respect to s and dividing by p, we obtain $\partial_- v_{ps}(f) = \partial_- v_s(f)$. Since the function $s \mapsto v_s(f)$ is concave, the function $s \mapsto \partial_- v_s(f)$ is nondecreasing; the above equality implies that it is constant. In other words, $s \mapsto v_s(f)$ is a linear function of s, which we can write as $v_s(f) = ns + r$ for some integer n and some real number r. The equality $v_{ps}(f) = pv_s(f)$ then implies that $r = 0$, so that $v_s(f) = ns$ for all $s > 0$ and therefore $|f|_\rho = \rho^n$ for all $0 < \rho < 1$. $\square$

Proof of Theorem 5. Let f be a nonzero element of $B^{\varphi=1}$. It follows from Lemma 6 and Corollary 2 that f belongs to the subring $\mathbf{A}_{\inf}[\frac{1}{p}] \subseteq B$. That is, f admits a unique Teichmüller expansion

$$f = \sum_{n \gg -\infty} [c_n] p^n,$$

where each c_n belongs to $\mathcal{O}_C^\flat$. We then have

$$\sum_{n \gg -\infty} [c_n] p^n = f = \varphi(f) = \sum_{n \gg -\infty} [c_n^p] p^n,$$

so that each coefficient c_n satisfies $c_n = c_n^p$ in the field $C^\flat$, and therefore belongs to the finite field $\mathbf{F}_p \subseteq C^\flat$. The equality $f = \sum_{n \gg -\infty} [c_n] p^n$ now shows that f belongs to $\mathbf{Q}_p = W(\mathbf{F}_p)[\frac{1}{p}]$, as desired. $\square$