Lecture 17: Algebraic Closure of Untilts

November 9, 2018

Our goal in this lecture is to prove the following result, which we have used several times without proof:

Theorem 1. *Let K be a perfectoid field. If the tilt $K^\flat$ is algebraically closed, then K is algebraically closed.*

We will prove Theorem 1 using an approximation argument which is similar to (but much easier than) the strategy of the last two lectures. The key point is to prove the following:

Proposition 2. *Let K be a perfectoid field such that the tilt $K^\flat$ is algebraically closed, and let $f(x) = x^n + a_1x^{n-1} + \cdots + a_n \in K[x]$ be a non-constant irreducible polynomial. Let y be an element of K. Then there exists an element $y' \in K$ satisfying*

$$|y - y'|_K \leq |f(y)|_K^{1/n} \qquad |f(y')|_K \leq |pf(y)|_K.$$

Proof of Theorem 1 from Proposition 2. Let K be a perfectoid field such that $K^\flat$ is algebraically closed. We assume that K has characteristic zero (otherwise there is nothing to prove). We wish to show that K is algebraically closed: that is, that every non-constant polynomial $f(x) \in K[x]$ has a root in K. Without loss of generality, we may assume that $f(x)$ is monic and irreducible of degree $n > 0$. Replacing $f(x)$ by $p^{nd}f(\frac{x}{p^d})$ for $d \gg 0$, we may assume that the coeffcients of f belong to $\mathcal{O}_K$. Setting $y_0 = 0$, it follows that $f(y_0) \in \mathcal{O}_K$, or equivalently that $|f(y_0)|_K \leq |p^0|_K$. Applying Proposition 2, we deduce that there exists $y_1 \in K$ satisfying $|y_0 - y_1|_K \leq |f(y_0)|_K^{1/n} \leq |p^0|_K^{1/n}$ and $|f(y_1)|_K \leq |pf(y_0)|_K \leq |p|_K$. Applying Proposition 2 to the element y_1, we obtain an element $y_2 \in \mathcal{O}_K$ satisfying $|y_1 - y_2|_K \leq |f(y_1)|_K^{1/n} \leq |p|_K^{1/n}$ and $|f(y_2)|_K \leq |pf(y_1)|_K \leq |p^2|_K$. Proceeding in this way, we obtain a sequence of elements $y_0 = 0, y_1, y_2, \ldots \in K$ satisfying

$$|y_m - y_{m+1}|_K \leq |p^m|_K^{1/n} \qquad |f(y_m)|_K \leq |p^m|_K.$$

It follows from the first inequality (and the completeness of K) that the sequence $\{y_m\}$ converges to an element $y \in K$. Then

$$|f(y)|_K = \lim_{m \to \infty} |f(y_m)|_K = 0,$$

so that y is a root of f. $\square$

For the proof of Proposition 2, we will use the following result from the theory of valued fields:

Theorem 3. *Let K be a field which is complete with respect to a non-archimedean absolute value $|\bullet|_K$, and let L be a finite extension field of K. Then $|\bullet|_K$ can be extended uniquely to an absolute value on the field $|\bullet|_L$.*

Remark 4. In the situation of Theorem 3, the absolute value $|\bullet|_L$ is given concretely by the formula

$$|x|_L = |N_{L/K}(x)|_K^{1/\deg(L/K)},$$

where $N_{L/K} : L \to K$ denotes the norm map and $\deg(L/K)$ denotes the degree of the field extension $K \hookrightarrow L$. To prove this, we are free to enlarge L and may thereby assume that L is a normal extension of K. In this case, we can write

$$N_{L/K}(x) = (\prod_{\gamma \in \mathrm{Gal}(L/K)} \gamma(x))^{d_0},$$

where d_0 is the inseparable degree of L over K. We therefore have

$$|N_{L/K}(x)|_K^{1/\deg(L/K)} = (\prod_{\gamma \in \mathrm{Gal}(L/K)} |\gamma(x)|_L)^{1/|\mathrm{Gal}(L/K)|}.$$

The desired identity then follows from formula $|x|_L = |\gamma(x)|_L$ for $\gamma \in \mathrm{Gal}(L/K)$ (by virtue of the uniqueness asserted in Theorem 3).

Warning 5. In the situation of Theorem 3, one cannot drop the assumption that K is complete. If K is not complete, then the norm $| \bullet |_K$ can generally be extended in many different ways to extension fields L over K, and the formula $|x|_L = |N_{L/K}(x)|_K^{1/\deg(L/K)}$ of Remark 4 need not define an absolute value on L.

Corollary 6. *Let K be a field which is complete with respect to a non-archimedean absolute value $| \bullet |_K$, and let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ be an irreducible polynomial with coefficients in K. If a_n belongs to $\mathcal{O}_K$, then each a_i belongs to $\mathcal{O}_K$.*

Proof. Let L be a finite normal extension of K over which the polynomial $f(x)$ factors as a product $f(x) = (x - r_1) \cdot (x - r_2) \cdots (x - r_n)$. Equip L with the absolute value $| \bullet |_L$ of Theorem 3. Since the roots r_i are conjugate by the action of the Galois group $\mathrm{Gal}(L/K)$, they must all have the same absolute value; that is, there exists a real number λ satisfying $|r_i|_L = \lambda$ for all i. Then $a_n = (-1)^n \prod_{i=1}^n r_i$. Consequently, if a_n belongs to $\mathcal{O}_K$, then each r_i belongs to $\mathcal{O}_L$. It follows that the polynomial

$$f(x) = \prod_{i=1}^n (x - r_i)$$

has coefficients in $\mathcal{O}_L$, so that each a_i belongs to $\mathcal{O}_L \cap K = \mathcal{O}_K$ as desired. □

Proof of Proposition 2. Let K be a perfectoid field such that the tilt $K^\flat$ is algebraically closed, and let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in K[x]$ be a non-constant irreducible polynomial. We wish to show that, for each element $y \in K$, we can find another point $y' \in K$ satisfying

$$|y - y'|_K \leq |f(y)|_K^{1/n} \qquad |f(y')|_K \leq |pf(y)|_K.$$

Replacing $f(x)$ by the polynomial $f(x + y)$, we can reduce to the case $y = 0$; in this case, we wish to show that there exists $y' \in K$ satisfying

$$|y'|_K \leq |f(0)|_K^{1/n} \qquad |f(y')|_K \leq |pf(0)|_K.$$

Let us assume that $f(0) \neq 0$ (otherwise, we can take $y' = 0$ and there is nothing to prove). Note that the value group of K is the same as the value group of $K^\flat$, and is therefore divisible (since $K^\flat$ is algebraically closed). We can therefore choose an element $c \in K$ satisfying $|c|_K = |f(0)|_K^{1/n}$. In this case, we can rewrite the inequalities above as

$$|\frac{y'}{c}|_K \leq 1 \qquad |\frac{1}{c^n} f(c \cdot \frac{y'}{c})|_K \leq |p|_K.$$

Replacing $f(x)$ by the monic polynomial $\frac{1}{c^n} f(cx)$ (and y' by $\frac{y'}{c}$), we can reduce to the case where $|f(0)|_K = 1$. In this case, we wish to show that there exists $y' \in K$ satisfying

$$|y'|_K \leq 1 \qquad |f(y')|_K \leq |p|_K.$$

Write $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$. Our assumption that $|f(0)|_K = 1$ guarantees that a_n belongs to $\mathcal{O}_K$. Applying Corollary 6, we see that each of the coefficients a_i belongs to $\mathcal{O}_K$. We can therefore choose elements $b_i \in \mathcal{O}_K^\flat$ satisfying $b_i^\sharp \equiv a_i \pmod{p}$. Set

$$g(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n \in K^\flat[x].$$

Since $K^\flat$ is algebraically closed, the polynomial $g(x)$ factors as a product

$$g(x) = (x - r_1) \cdots (x - r_n)$$

for some $r_1, r_2, \ldots, r_n \in K^\flat$. Note that we have

$$|r_1|_{K^\flat} \cdots |r_n|_{K^\flat} = |(-1)^n b_n|_{K^\flat} \leq 1.$$

It follows that there must exist $r \in \{r_1, \ldots, r_n\}$ satisfying $|r|_{K^\flat} \leq 1$, so that r belongs to $\mathcal{O}_K^\flat$. Setting $y' = r^\sharp$, we have $|y'|_K = |r|_{K^\flat} \leq 1$, and

$$\begin{aligned} f(y') &= y'^n + a_1 y'^{n-1} + \cdots + a_n \\ &\equiv y'^n + b_1^\sharp y'^{n-1} + \cdots + b_n^\sharp \pmod{p} \\ &= (r^\sharp)^n + b_1^\sharp (r^\sharp)^{n-1} + \cdots + b_n^\sharp \\ &\equiv (g(r))^\sharp \pmod{p} \\ &= 0 \end{aligned}$$

so that $|f(y')|_K \leq |p|_K$, as desired. $\square$