Lecture 16: Converging to a Zero

November 8, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Let f be an element of the ring $\mathbf{A}_{\inf}$ which is primitive of degree d: that is, an element which admits a Teichmüller expansion $\sum_{n \geq 0} [c_n] p^n$ satisfying

$$c_0 \neq 0 \qquad |c_i|_{C^\flat} < 1 \text{ for i ¡ d} \qquad |c_d|_{C^\flat} = 1.$$

Assume that $d > 0$, and let $\lambda \in (0,1)$ be the largest element for which the function $s \mapsto v_s(f)$ fails to be differentiable at $-\log(\lambda)$; that is, λ satisfies

$$|c_i|\lambda^i \leq \lambda^d \text{ for all } i$$

$$|c_i|\lambda^i = \lambda^d \text{ for some } i < d$$

Our goal in this lecture is to complete the proof of the following result:

Proposition 1. *Then there exists a point $y \in Y$ satisfying $d(0,y) = \lambda$ and $f(y) = 0$.*

Note that we have $|f|_\lambda = \lambda^d$. Consequently, for each point $y \in Y$ satisfying $d(0,y) = \lambda$, we automatically have

$$|f(y)| \leq |f|_\lambda = \lambda^d.$$

Moreover, we expect the inequality to be strict if and only if y is "close" to a root of f. More precisely, if f factors as a product of distinguished elements of $\mathbf{A}_{\inf}$ (which will follow once Proposition 1 has been proved), then we expect

$$|f(y)| = \lambda^d \cdot \prod \frac{d(y', y)}{\lambda},$$

where the product is taken over the collection of all y' satisfying $d(0, y') = \lambda$ and $f(y') = 0$ (counted with multiplicity!); here at most d factors appear. In particular, we should be able to choose at least one such point y' satisfying

$$\frac{d(y', y)}{\lambda} \leq (\frac{|f(y)|}{\lambda^d})^{1/d}.$$

We now show that this is the case.

Lemma 2. *Let y be a point of Y satisfying $d(0,y) = \lambda$, and suppose that $|f(y)| = \lambda^d \cdot \alpha$ for some $\alpha < 1$. Then there exists a point $y' \in Y$ satisfying $d(y, y') \leq \lambda \cdot \alpha^{1/d}$ and $f(y') \leq \lambda^{d+1} \cdot \alpha$.*

Proof of Proposition 1 from Lemma 2. We proved in Lecture 15 that there exists a point $y_1 \in Y$ satisfying $d(0, y_1) = \lambda$ and $|f(y_1)| \leq \lambda^{d+1}$. Applying Lemma 2, we can choose a point $y_2 \in Y$ satisfying $d(y_1, y_2) \leq \lambda^{1+\frac{1}{d}}$ and $|f(y_2)| \leq \lambda^{d+2}$. Note that we then also have $d(0, y_2) = \lambda$, so we can apply Lemma 2 again to choose a point $y_3 \in Y$ satisfying $d(y_2, y_3) \leq \lambda^{1+\frac{2}{d}}$ and $|f(y_3)| \leq \lambda^{d+3}$. Continuing in this way, we obtain a sequence of points $\{y_n\}$ on the circle $Y_{[\lambda,\lambda]}$ satisfying

$$d(y_n, y_{n+1}) \leq \lambda^{1+\frac{n}{d}} \qquad |f(y_n)| \leq \lambda^{d+n}.$$

The first inequality implies that the sequence $\{y_n\}$ is Cauchy, and therefore converges to a point $y \in Y_{[\lambda,\lambda]}$. The second inequality implies that $|f(y)| = \lim_{n\to\infty} |f(y_n)| = 0$, so that $f(y) = 0$. □

Proof of Lemma 2. Fix a point $y = (K, \iota) \in Y$ satisfying $d(0, y) = \lambda$ and $|f(y)|_K \leq \lambda^d \cdot \alpha$. Let ξ be a distinguished element of $\mathbf{A}_{\inf}$ satisfying $\xi(y) = 0$. Since $\mathbf{A}_{\inf}$ is ξ-adically complete and every element of $\mathbf{A}_{\inf}/\xi$ belongs to the image of $\sharp : \mathcal{O}_C^\flat \to \mathcal{O}_K$, we can write f as a sum

$$\sum_{n\geq 0} [c_n]\xi^n$$

(beware that this representation is *not* unique, because the map $\sharp : \mathcal{O}_C^\flat \to \mathcal{O}_K$ is not bijective). Note that under the reduction map

$$\mathbf{A}_{\inf} = W(\mathcal{O}_C^\flat) \to W(\mathcal{O}_C^\flat / \mathfrak{m}_C^\flat) = W(k),$$

the image of ξ is a unit multiple of p (since ξ is distinguished) and the image of f is a unit multiple of p^d (since f is primitive of degree d). It follows that $|c_i|_{C^\flat} < 1$ for $i < d$ and that $|c_d|_{C^\flat} = 1$. Replacing f by $f/[c_d]$, we may assume without loss of generality that $c_d = 1$. Note that we have

$$|c_0|_{C^\flat} = |[c_0](y)|_K = |f(y)|_K = \lambda^d \cdot \alpha.$$

We will assume that $c_0 \neq 0$ (otherwise, we can take $y' = y$).

Consider the polynomial

$$F(x) = c_0 + c_1 x + \cdots + c_{d-1}x^{d-1} + x^d \in C_\flat[x].$$

Since $C^\flat$ is algebraically closed, we can factor $F(x)$ as a product of linear factors

$$F(x) = (x - r_1)(x - r_2)\cdots(x - r_d)$$

for some elements $r_1, r_2, \ldots, r_d \in C^\flat$. Choose $r \in \{r_1, \ldots, r_d\}$ so that the absolute value of r is as small as possible. Note that, for $0 \leq m \leq d$, we have $c_m = \pm e_{d-m}(r_1, \ldots, r_d)$, where e_{d-m} denotes the $(d-m)$th elementary symmetric polynomial. We therefore have

$$\begin{aligned} |r|_{C^\flat}^m |c_m|_{C^\flat} &\leq |r|_{C^\flat}^m \sup_{J\subseteq\{1,\ldots,d\},|J|=d-m} \prod_{j\in J} |r_j|_{C^\flat} \\ &\leq \frac{\prod_{j=1}^d |r_j|_{C^\flat}}{|r|^m} \\ &= |c_0|_{C^\flat} \\ &= \lambda^d \cdot \alpha. \end{aligned}$$

In the special case $m = d$, we have $|r|_{C^\flat}^d \leq \lambda^d \cdot \alpha$, or $|r|_{C^\flat} \leq \lambda \cdot \alpha^{1/d}$. Set $\xi' = \xi - [r]$. Then ξ' is also a distinguished element, vanishing at a point $y' \in Y$. We have

$$d(y, y') = |\xi'(y)|_K = |-[r](y)|_K = |r|_{C^\flat} \leq \lambda \cdot \alpha^{1/d}.$$

It follows that $d(y, y') < \lambda = d(0, y)$, so that we have $d(0, y') = \lambda$. Let K' be the untilt of $C^\flat$ corresponding to the point y' and let $\sharp : C^\flat \to K'$ be the usual map (given by $x^\sharp = [x](y')$). Then $\xi(y') = (\xi' + [r])(y') = r^\sharp$. We therefore have

$$\begin{aligned} \frac{f(y')}{[c_0]} &= \sum_{n\geq 0} \frac{c_n^\sharp}{c_0^\sharp} \xi(y')^n \\ &= \sum_{n\geq 0} (\frac{c_n r^n}{c_0})^\sharp. \end{aligned}$$

Note that the ration $\frac{c_n r^n}{c_0}$ belongs to $\mathcal{O}_{C^\flat}$ for $n \leq d$ (by virtue of the inequality $|c_n r^n|_{C^\flat} \leq |c_0|$ established above). For $n > d$, we have

$$|\frac{c_n r^n}{c_0}|_{C^\flat} \leq |\frac{r^n}{c_0}|_{C^\flat} \leq \frac{|r|^n_{C^\flat}}{\lambda^d \cdot \alpha} \leq \frac{\lambda^n \cdot \alpha^{n/d}}{\lambda^d \cdot \alpha} \leq \lambda = |p|_{K'}.$$

It follows that each $(\frac{c_n r^n}{c_0})^\sharp$ belongs to the valuation ring $\mathcal{O}^\flat_K$, and is divisible by p (in $\mathcal{O}^\flat_K$) when $n > d$. We therefore compute

$$\begin{aligned}
\frac{f(y')}{[c_0]} &= \sum_{n \geq 0} (\frac{c_n r^n}{c_0})^\sharp \\
&\equiv \sum_{n=0}^{d} (\frac{c_n r^n}{c_0})^\sharp \pmod p \\
&\equiv (\sum_{n=0}^{d} \frac{c_n r^n}{c_0})^\sharp \pmod p \\
&= \frac{F(r)^\sharp}{c_0^\sharp} \\
&= 0.
\end{aligned}$$

We therefore have

$$|f(y')|_{K'} \leq |[c_0]|_{K'} \cdot |p|_{K'} = \lambda^d \cdot \alpha \cdot \lambda = \lambda^{d+1} \cdot \alpha.$$

□