Lecture 15: Zeroes of Primitive Elements

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Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Our goal is to show if f is a nonzero element of the ring $B_{[\rho,\rho]}$ which is not invertible, then f vanishes at some point y lying on the "circle"

$$Y_{[\rho,\rho]} = \{y = (K, \iota) \in Y : |p|_K = \rho\}.$$

In the last lecture, we said that an element f of $B_{[\rho,\rho]}$ is *good* if we can write

$$f = g \cdot \xi_1 \cdot \xi_2 \cdot \cdots \cdot \xi_n,$$

where g is invertible and each ξ_i is a distinguished element of $\mathbf{A}_{\text{inf}}$ vanishing at some point $y_i \in Y_{[\rho,\rho]}$. Moreover, we proved that if f is a nonzero, noninvertible element of $B_{[\rho,\rho]}$ which can be realized as the limit $\varinjlim_{i\to\infty} f_i$ (with respect to the Gauss norm $|\bullet|_\rho$ where each f_i is good, then f has a zero. Note that any element of $B_{[\rho,\rho]}$ can be approximated arbitrarily well by elements of the ring $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$, which admit Teichmüller expansions

$$\sum_{n \gg -\infty} [c_n] p^n.$$

Moreover, every such sum can be approximated arbitrarily well by elements with *finite* Teichmüller expansions. It will therefore suffice to prove the following:

Proposition 1. *Let f be an element of $B_{[\rho,\rho]}$ which admits a finite Teichmüller expansion $\sum_{n=-N}^{N} [c_n] p^n$. Then f is good.*

It will be convenient to introduce a bit of terminology.

Definition 2. Let f be an element of $\mathbf{A}_{\text{inf}}$, given by a Teichmüller expansion $\sum_{n\geq 0} [c_n] p^n$. We say that f is *primitive* if $c_0 \neq 0$ and $|c_d|_{C^\flat} = 1$ for some integer d. If d is the smallest such integer, then we say that f is *primitive of degree d*.

Remark 3. Let f be an element of $\mathbf{A}_{\text{inf}}$. Then f is primitive if and only if it satisfies the following conditions:

- The element f is not divisible by p.
- The element f has nonzero image $\overline{f}$ under the map

  $$\mathbf{A}_{\text{inf}} = W(\mathcal{O}_C^\flat) \to W(\mathcal{O}_C^\flat / \mathfrak{m}_C^\flat) = W(k),$$

  where k denotes the residue field of C. In this case, the degree d of f is characterized by the equality of ideals $(p^d) = (\overline{f})$ in $W(k)$.

It follows that if $f = gh$ is primitive, then g and h are also primitive, with $\deg(f) = \deg(g) + \deg(h)$.

Remark 4. An element $f \in \mathbf{A}_{\text{inf}}$ is primitive of degree 1 if and only if it is distinguished and corresponds to a *characteristic zero* untilt of $C^\flat$.

Remark 5. Let f be an element of $B_{[\rho,\rho]}$ which admits a finite Teichmüller expansion $\sum_{n=-N}^{N}[c_n]p^n$. Then we can write $f = p^m \cdot [c] \cdot g$, where m is an integer, c is a nonzero element of $C^\flat$, and $g \in \mathbf{A}_{\text{inf}}$ is primitive. Moreover, the integer m and the absolute value $|c|_{C^\flat}$ are uniquely determined.

Exercise 6. Let $f = \sum_{n \gg -\infty}[c_n]p^n$ be an element of $\mathbf{A}_{\text{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. Show that the following conditions are equivalent:

- The element f is nonzero and the supremum $\sup_n\{|c_n|_{C^\flat}\}$ is actually achieved for some integer n.
- The function $s \mapsto v_s(f)$ fails to be differentiable at only finitely many points.
- The element f factors as a product $p^m \cdot [c] \cdot g$, where $g \in \mathbf{A}_{\text{inf}}$ is primitive.

Example 7. Let ϵ be an element of $1 + \mathfrak{m}_C^\flat$. Then the element $[\epsilon] - 1 \in \mathbf{A}_{\text{inf}}$ is not primitive, since it has vanishing image in $W(k)$. More explicitly, the problem is that $[\epsilon] - 1$ has infinitely many zeroes in Y: we have

$$\begin{aligned}
[\epsilon] - 1 &= \frac{[\epsilon] - 1}{[\epsilon^{1/p}] - 1} \cdot ([\epsilon^{1/p}] - 1) \\
&= \frac{[\epsilon] - 1}{[\epsilon^{1/p}] - 1} \cdot \frac{[\epsilon^{1/p}] - 1}{[\epsilon^{1/p^2}] - 1} \cdot ([\epsilon^{1/p^2}] - 1) \\
&= \cdots
\end{aligned}$$

We will deduce Proposition 1 from the following:

Proposition 8. *Let f be an element of $\mathbf{A}_{\text{inf}}$ which is primitive of degree $d > 0$. Then f admits a factorization*

$$f = \xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_d,$$

where each ξ_i is a distinguished element vanishing at some point $y_i \in Y$.

Proof of Proposition 1 from Proposition 8. Let f be an element of $B_{[\rho,\rho]}$ which admits a finite Teichmüller expansion. By Remark 5, we can write $f = p^m \cdot [c] \cdot g$, where g is a primitive element of $\mathbf{A}_{\text{inf}}$. Let d be the degree of g. If $d = 0$, then g is invertible in $\mathbf{A}_{\text{inf}}$ and therefore f is invertible in $B_{[\rho,\rho]}$. Otherwise, we can use Proposition 8 to write $g = \xi_1 \cdot \ldots \cdot \xi_d$, where each ξ_i is a distinguished element of $\mathbf{A}_{\text{inf}}$ vanishing at some point $y_i \in Y$. Rearranging the product if necessary, we may assume that $y_1, y_2, \cdots, y_m$ belong to the circle $Y_{[\rho,\rho]}$, and that $y_i \notin Y_{[\rho,\rho]}$ for $m < i \leq d$. Then ξ_i is invertible in $B_{[\rho,\rho]}$ for $m < i \leq d$ (this follows from Lecture 13, but is also easy to see directly). The factorization

$$f = (p^m \cdot [c] \cdot \prod_{i=m+1}^{d} \xi_i) \cdot \xi_1 \cdot \cdots \cdot \xi_m$$

now shows that f is good. □

Let $f = \sum_{n \geq 0}[c_n]p^n$ be an element of $\mathbf{A}_{\text{inf}}$ which is primitive of degree d, and consider the function

$$v_\bullet(f) : \mathbf{R}_{>0} \to \mathbf{R} \qquad s \mapsto v_s(f) = \inf_n (v(c_n) + ns).$$

Note that for $n \geq d$, we have $v(c_n) + ns > v(c_d) + ds = ds$, so we might as well only take the infimum only over the set $\{0, 1, \ldots, d\}$. For s sufficiently small, this infimum is realized when $n = d$ and we have $v_s(f) = ds$. When s is sufficiently large, the infimum is realized when $n = 0$ and we have $v_s(f) = v(c_0)$. It follows that, if $d \neq 0$, then there is some smallest positive real number s such that the function $v_\bullet(f)$ is not differentiable at s: that is, for which $\partial_- v_s(f) > \partial_+ v_s(f)$. Write $s = -\log(\lambda)$ for $\lambda \in (0, 1)$. We will prove the following:

Proposition 9. *Let f be an element of $\mathbf{A}_{\inf}$ which is primitive of degree $d > 0$, and let $\lambda \in (0,1)$ be defined as above. Then there exists a point $y \in Y_{[\lambda,\lambda]}$ satisfying $f(y) = 0$.*

Proof of Proposition 8 from Proposition 9. Let f be an element of $\mathbf{A}_{\inf}$ which is primitive of degree d; we wish to show that f can be written as a product of d distinguished elements (corresponding to characteristic zero untilts of $C^{\flat}$). We proceed by induction on d; the case $d = 1$ is immediate from Remark 4. To carry out the inductive hypothesis, we observe that Propositino 9 guarantees that there is a point $y \in Y$ satisfying $f(y) = 0$, so that f factors as a product $f = g \cdot \xi$, where ξ is a distinguished element vanishing at y. Remark 3 then shows that g is primitive of degree $d-1$ and can therefore be factored as a product of $d-1$ distinguished elements by our inductive hypothesis. $\square$

Remark 10. To prove Proposition 8 from Proposition 9, the equality $d(y, 0) = \lambda$ is irrelevant. We include it in the statement of Proposition 9 to highlight that the overall strategy is a bit subtle. Let f be a nonzero element of $B_{[\rho,\rho]}$ which is not invertible. Then f is *a priori* defined *only* at points belonging to the circle $Y_{[\rho,\rho]}$, and we wish to show that there is a point $y \in Y_{[\rho,\rho]}$ satisfying $f(y) = 0$. To find this point, we are writing f as the limit $\varinjlim f_n$ of elements which admit finite Teichmüller expansions, and can therefore be evaluated at *any* point of Y. For $n \gg 0$, we expect that the functions f_n must also vanish at some point $y_n \in Y_{[\rho,\rho]}$, and the argument of Lecture 14 shows that we can choose these points so that the sequence $\{y_n\}$ converges to a point $y \in Y_{[\rho,\rho]}$ where f vanishes. However, Proposition 9 does not produce the points y_n directly. Each f_n vanishes at finitely many points of Y, some of which lie on the circle $Y_{[\rho,\rho]}$ and some of which do not. The proof of Proposition 9 will actually select a zero of f_n that is *furthest from the origin*. In order to find the desired zero lying on the circle $Y_{[\rho,\rho]}$, we actually need to apply Proposition 9 repeatedly (to primitive elements of $\mathbf{A}_{\inf}$ which are factors of $p^m \cdot [c] \cdot f_n$)

Let us now begin the proof of Proposition 9. Note that we have $v_s(f) = ds$ for $s \in (0, -\log(\lambda)]$, In particular, we have $v_{-\log(\lambda)}(f) = -d\log(\lambda)$, or equivalently $|f|_\lambda = \lambda^d$. It follows that for any point $y = (K_y, \iota)$ lying on the circle $Y_{[\lambda,\lambda]}$, we have $|f(y)|_{K_y} \leq |f|_\lambda = \lambda^d$. We saw in the previous lecture that if f is good (when regarded as an element of $B_{[\lambda,\lambda]}$), then the equality is strict if and only if y is "close" to a point where f vanishes: that is, if and only if there is a point $y' \in Y_{[\lambda,\lambda]}$ satisfying $d(y, y') < \lambda$ and $f(y') = 0$. Of course, we do not yet know that f is good (that's a special case of what we are trying to prove). But it suggests that if we can find a point $y \in Y_{[\lambda,\lambda]}$ with $|f(y)|_{K_y} < \lambda^d$, then we will be on the right track. We therefore begin by proving the following weaker version of Proposition 9 (we will complete the proof in the next lecture).

Lemma 11. *Let f be an element of $\mathbf{A}_{\inf}$ which is primitive of degree $d > 0$, and let $\lambda \in (0,1)$ be defined as above. Then there exists a point $y \in Y_{[\lambda,\lambda]}$ satisfying $|f(y)|_{K_y} \leq \lambda^{d+1}$.*

Proof. Write $f = \sum_{n \geq 0} [c_n] p^n$. Multiplying f by $[c_d^{-1}]$ if necessary, we may assume without loss of generality that $c_d = 1$. By our choice of λ, we have

$$|c_i|_{C^\flat} \lambda^i \leq \lambda^d$$
$$|c_i|_{C^\flat} \leq \lambda^{d-i}$$

and that equality holds for at least one value of i.

Consider the polynomial

$$F(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0 \in C^\flat[x].$$

Since $C^\flat$ is algebraically closed, this polynomial factors as a product of linear factors: that is, we can find elements $r_1, \ldots, r_d \in C^\flat$ satisfying $F(x) = (x - r_1)(x - r_2)\cdots(x - r_d)$. Let λ' denote the largest of the absolute values of these roots (we will see in a moment that $\lambda' = \lambda$). Without loss of generality, we may assume that

$$|r_i|_{C^\flat} = \lambda \text{ for } i = 1, \ldots, m \qquad |r_i|_{C^\flat} < \lambda \text{ for } i = m+1, m+2, \ldots, d.$$

Let $e_m(r_1, \dots, r_d)$ denote the mth symmetric function of r_1 through r_d. Then

$$e_m(r_1, \dots, r_d) = r_1 \cdots r_m + \text{ terms of absolute value } < \lambda'^m \ .$$

We therefore have

$$\lambda'^m = |e_m(r_1, \dots, r_d)|_{C^\flat} = |c_{n-m}|_{C^\flat} \leq \lambda^m.$$

On the other hand, there is some integer $0 \leq i < d$ satisfying

$$\lambda^{d-i} = |c_i|_{C^\flat} = |e_{d-i}(r_1, \dots, r_d)|_{C^\flat} \leq \lambda'^{d-i}.$$

Combining these, we obtain $\lambda = \lambda'$.

Set $r = r_1$, and note that $c_i = \pm e_{d-i}(r_1, \dots, r_n)$ is divisible by r^{d-i} for $0 \leq i \leq d$. Set $\xi = p - [r]$. Then ξ is a distinguished element of $\mathbf{A}_{\text{inf}}$ vanishing at a point $y \in Y$ satisfying $d(0, y) = |r|_{C^\flat} = \lambda' = \lambda$. Let K denote the corresponding untilt of $C^\flat$ and $\theta : \mathbf{A}_{\text{inf}} \twoheadrightarrow \mathcal{O}_K$ associated quotient map. Then

$$\begin{aligned}
p^{-d} f(y) &= p^{-d}\theta(f) \\
&= \sum_{n \geq 0} c_n^\sharp p^{n-d} \\
&\equiv \sum_{i=0}^{d} (\frac{c_i}{r^{d-i}})^\sharp \pmod p \\
&\equiv (\sum_{i=0}^{d} \frac{c_i}{r^{d-i}})^\sharp \pmod p \\
&= (r^{-d} F(r))^\sharp \\
&= 0.
\end{aligned}$$

In other words, we have $f(y) \equiv 0 \pmod{p^{d+1}}$ in $\mathcal{O}_K$, which is equivalent to the desired inequality $|f(y)|_K \leq \lambda^{d+1}$. □