Lecture 14: The Metric Structure of Y

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Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Fix real numbers $0 < a \leq b < 1$ and set $\alpha = -\log(a)$, $\beta = -\log(b)$. Recall that our goal is to prove the following:

Theorem 1 (Existence of Roots, Version 1)**.** *Let f be a nonzero element of $B_{[a,b]}$, and suppose that $\partial_- v_\beta(f) \neq \partial_+ v_\alpha(f)$. Then there is a point $y \in Y_{[a,b]}$ such that $f(y) = 0$.*

Note that, in the situation of Theorem 1, we can find some $\rho \in [a, b]$ such that $\partial_- v_s(f) \neq \partial_+ v_s(f)$ for $s = -\log(\rho)$. To prove that Theorem 1 has a root in $Y_{[a,b]}$, it suffices to show that it has a root in $Y_{[\rho,\rho]}$. That is, we may assume without loss of generality that $a = \rho = b$. It will therefore suffice to prove the following special case of Theorem 1:

Theorem 2 (Existence of Roots, Version 2)**.** *Let f be a nonzero element of $B_{[\rho,\rho]}$ and set $s = -\log(\rho)$. If $\partial_- v_s(f) > \partial_+ v_s(f)$, then f vanishes at some point $y \in Y_{[\rho,\rho]}$.*

Note that, using the arguments of Lectures 12 and 13, Theorem 2 is equivalent to the following apparently stronger statement:

Corollary 3. *Let f be a nonzero element of $B_{[\rho,\rho]}$. Then f admits a factorization*

$$f = g \cdot \xi_1 \cdot \xi_2 \cdot \cdots \cdot \xi_n,$$

where each ξ_i is a distinguished element of $\mathbf{A}_{\text{inf}}$ vanishing at some point $y_i \in Y_{[\rho,\rho]}$, and g is an invertible element of $B_{[\rho,\rho]}$.

Here the hypothesis that $C^\flat$ is algebraically closed is essential: if $C^\flat$ is not algebraically closed, then the function $s \mapsto v_s(f)$ can fail to be differentiable due to "zeroes" coming from untilts of finite extension fields of $C^\flat$, rather than of $C^\flat$ itself. To say that $C^\flat$ is algebraically closed is to say that any *polynomial* equation in $C^\flat$ has a solution. To get from there to solving "analytic" equations like $f(y) = 0$, we will need to make some approximation arguments.

Notation 4. Let $\overline{Y}$ denote the set of isomorphism classes of untilts of $C^\flat$. We write $\overline{Y} = Y \cup \{0\}$, where Y is the set of isomorphism classes of characteristic zero untilts of $C^\flat$ and 0 denotes the isomorphism class of the characteristic p untilt (given by $C^\flat$ itself). For each point $y \in \overline{Y}$, we let ξ_y denote a distinguished element of $\mathbf{A}_{\text{inf}}$ which vanishes at y (so ξ_y is determined up to multiplication by a unit in $\mathbf{A}_{\text{inf}}$); for example, we can take $\xi_0 = p$.

For every pair of points $x, y \in \overline{Y}$, we let $d(x, y)$ denote the absolute value $|\xi_x(y)|_K$, where $y = (K, \iota)$. We will refer to $d(x, y)$ as the *distance from x to y*. Note that this quantity does not depend on the choice of distinguished element ξ_x: if ξ'_x is another distinguished element of $\mathbf{A}_{\text{inf}}$ vanishing at x, then $\xi_x(y)$ and $\xi'_x(y)$ differ by multiplication by a unit in $\mathcal{O}_K$, and therefore have the same absolute value in K.

Example 5. For $y = (K, \iota) \in \overline{Y}$, we have $d(0, y) = |p|_K$; this is the "distance from the origin" that we introduced earlier.

Proposition 6. *The function* $d : \overline{Y} \times \overline{Y} \to \mathbf{R}_{\geq 0}$ *is an ultrametric. That is, we have*

$$d(x, y) = 0 \Leftrightarrow x = y$$

$$d(x, y) = d(y, x)$$

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

Proof. Note that $d(x, y) = 0$ if and only if the distinguished element ξ_x vanishes at y, which holds if and only if $x = y$.

Fix any pair of points $x, y \in \overline{Y}$, corresponding to untilts K_x and K_y of $C^\flat$. Since $C^\flat$ is algebraically closed, we can write $\xi_x(y) = c^\sharp$ for some $c \in C^\flat$. Then c belongs to the maximal ideal $\mathfrak{m}_C^\flat$, so that ξ_x and $\xi_x - [c]$ have the same image under the map

$$\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to W(\mathcal{O}_C^\flat / \mathfrak{m}_C^\flat) = W(k).$$

It follows that $\xi_x - [c]$ is also a distinguished element of $\mathbf{A}_{\mathrm{inf}}$ which vanishes at the point y. We may therefore assume without loss of generality that $\xi_y = \xi_x - [c]$, so that

$$d(y, x) = |\xi_y(x)|_{K_x} = |\xi_x(x) - c^\sharp|_{K_x} = |c|_{C^\flat} = |c^\sharp|_{K_y} = |\xi_x(y)|_{C^\flat} = d(x, y);$$

here we write $c^\sharp$ both for the image of $[c]$ in K_x and its image in K_y.

To prove the third assertion, suppose we are given a point $z \in \overline{Y}$ corresponding to an untilt K_z. We then have

$$d(x, z) = |\xi_x(z)|_{K_z} = |\xi_y(z) + c^\sharp|_{K_z} \leq \max(|\xi_y(z)|_{K_z}, |c^\sharp|_{K_z}) = \max(d(y, z), d(x, y))$$

(where this time $c^\sharp$ denotes the image of $[c]$ in K_z). □

Proposition 7. *The set* $\overline{Y}$ *is* complete *with respect to the metric* $d(x, y)$.

Proof. Suppose we are given a Cauchy sequence $y_0, y_1, y_2, \ldots \in \overline{Y}$. Let ξ_{y_0} be a distinguished element of $\mathbf{A}_{\mathrm{inf}}$ which vanishes at y_0. Arguing as in the proof of Proposition 6, we can choose a sequence distinguished elements ξ_{y_n} vanishing at the points y_n, such that

$$\xi_{y_n} = \xi_{y_{n-1}} + [c_n],$$

where c_n is an element of $C^\flat$ satisfying $|c_n|_{C^\flat} = d(y_{n-1}, y_n)$.

Let $\pi \in C^\flat$ be a pseudo-uniformizer. Since the sequence $\{y_n\}$ is Cauchy, the sum

$$\sum_{n>0} [c_n]$$

converges with respect to the $[\pi]$-adic topology on $\mathbf{A}_{\mathrm{inf}}$ (recall that $\mathbf{A}_{\mathrm{inf}}$ is $[\pi]$-adically complete, since it is p-adically complete and p-torsion free and $\mathcal{O}_C^\flat = \mathbf{A}_{\mathrm{inf}}/(p)$ is π-adically complete and π-torsion free). Set $\xi = \xi_{y_0} + \sum_{n>0}[c_n]$. Then ξ and ξ_{y_0} have the same image under the map

$$\mathbf{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to W(\mathcal{O}_C^\flat / \mathfrak{m}_C^\flat) = W(k),$$

so ξ is a distinguished element of $\mathbf{A}_{\mathrm{inf}}$ vanishing at some point $y \in \overline{Y}$. We then compute

$$d(y, y_m) = |\xi(y_m)|_{K_{y_m}} = |\xi_m(y_m) + \sum_{n>m} c_n^\sharp|_{K_{y_m}} \leq \max\{|c_n|_{C^\flat}\}_{n>m},$$

which tends to zero as $m \to \infty$. It follows that the Cauchy sequence $\{y_n\}$ converges to y. □

Let us now return to the situation of Theorem 2. Fix $0 < \rho < 1$, and let f be a nonzero element of $B_{[\rho,\rho]}$. Recall that, if $y = (K, \iota)$ is a point of Y satisfying $d(0, y) = |p|_K = \rho$, then we have $|f(y)|_K \leq |f|_\rho$. In general, this inequality is strict. However, if f is an *invertible* element of $B_{[\rho,\rho]}$, then we also have

$$|(\frac{1}{f})(y)|_K \leq |\frac{1}{f}|_\rho = \frac{1}{|f|_\rho},$$

which implies that $|f(y)|_K = |f|_\rho$.

Let us say that an element f of $B_{[\rho,\rho]}$ is *good* if it satisfies the conclusion of Corollary 3: that is, if f admits a factorization

$$f = g \cdot \xi_1 \cdot \xi_2 \cdot \cdots \cdot \xi_n,$$

where g is an invertible element of $B_{[\rho,\rho]}$ and each ξ_i is a distinguished element vanishing at some point $y_i \in Y_{[a,b]}$. We then compute

$$\begin{aligned}
|f(y)|_K &= |g(y)|_K \cdot |\xi_1(y)|_K \cdot \cdots \cdot |\xi_n(y)|_K \\
&= |g|_\rho \cdot \prod_{i=1}^{n} d(y_i, y) \\
&= \frac{|f|_\rho}{\prod_{i=1}^{n} |\xi_i|_\rho} \prod_{i=1}^{n} d(y_i, y) \\
&= |f|_\rho \prod_{i=1}^{n} \frac{d(y_i, y)}{\rho}.
\end{aligned}$$

In other words, in particular, we see that the equality $|f(y)|_K = |f|_\rho$ holds in the *generic case* where y is at distance ρ from each of the zeroes y_i of the function f. However, we have a strict inequality whenever $d(y_i, y) < \rho$ for some i.

Proposition 8. *Let f be a good element of $B_{[\rho,\rho]}$ having n zeroes in $Y_{[\rho,\rho]}$ (counted with multiplicity), and let g be any nonzero element of $B_{[\rho,\rho]}$. Suppose that $|f - g|_\rho < |f|_\rho$. Then, for any point $y = (K, \iota) \in Y_{[\rho,\rho]}$ satisfying $g(y) = 0$, there exists a point $y' \in B_{[\rho,\rho]}$ satisfying $f(y') = 0$ and $d(y, y') < \rho(\frac{|f-g|_\rho}{|f|_\rho})^{1/n}$.*

Proof. Let $y_1, \ldots, y_n$ be the zeroes of f (counted with multiplicity). If $g(y) = 0$, we have

$$\begin{aligned}
|f - g|_\rho &\geq |(f - g)(y)|_K \\
&= |f(y)|_K \\
&= |f|_\rho \prod_{i=1}^{n} \frac{d(y_i, y)}{\rho}.
\end{aligned}$$

It follows that at least one of the factors $\frac{d(y_i,y)}{\rho}$ must be less than or equal to $(\frac{|f-g|_\rho}{|f|_\rho})^{1/n}$. □

Corollary 9. *Let f be a nonzero element of $B_{[\rho,\rho]}$ which is given as the limit of a Cauchy sequence $\{f_i\}$ with respect to the Gauss norm $| \bullet |_\rho$. Suppose that each f_i is good. If $\partial_- v_s(f) > \partial_+ v_s(f)$ for $s = -\log(\rho)$, then f vanishes at some point in $Y_{[\rho,\rho]}$.*

Proof. Passing to a subsequence, we may assume that

$$v_s(f) = v_s(f_i) \qquad \partial_- v_s(f) = \partial_- v_s(f_i) \qquad \partial_+ v_s(f) = \partial_+ v_s(f_i)$$

$$|f_{i+1} - f_i|_\rho < |f|_\rho$$

for all i. Set $n = \partial_- v_s(f) - \partial_+ v_s(f) > 0$. Then each f_i has exactly n zeroes in $Y_{[\rho,\rho]}$, counted with multiplicity. Applying Proposition 8, we can choose a sequence $\{y_i\}$ in $Y_{[\rho,\rho]}$ such that $f_i(y_i) = 0$ and

$$d(y_{i+1}, y_i) \leq \rho(\frac{|f_{i+1} - f_i|_\rho}{|f|_\rho})^{1/n}$$

It follows that the sequence $\{y_i\}$ is Cauchy and therefore converges to some point $y \in \overline{Y}$ (Proposition 7). We then have

$$|f_i(y)|_K \leq |f_i|_\rho \cdot \frac{d(y_i, y)}{\rho} = |f|_\rho \cdot \frac{d(y_i, y)}{\rho} \to 0$$

as $i \to \infty$, so $f(y) = \lim_{i\to\infty} f_i(y)$ vanishes in K. $\square$