Lecture 13: Digression-Jensen's Formula

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The constructions of the previous lecture(s) have an analogue in complex analysis.Let a and b be positive real numbers, and let f be a holomorphic function defined on the annulus $\{z \in \mathbf{C} : a < |z| < b\}$ which is not identically zero. Define a function

$$A(\bullet, f) : (\log(a), \log(b)) \to \mathbf{R} \qquad A(s, f) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\exp(s + i\theta))| d\theta.$$

In other words, $A(s, f)$ is the average value of $|\log f(z)|$ on the circle $\{z \in \mathbf{C} : |z| = \exp(s)\}$ of radius $\exp(s)$.

Exercise 1. Show that the average $A(s, f)$ is well-defined even when f vanishes at some points of the circle $\{z \in \mathbf{C} : |z| = \exp(s)\}$: that is, the function $\theta \mapsto \log f(\exp(s + i\theta))$ is always integrable, even when it fails to be well-defined at finitely many points. Moreover, if the holomorphic function f is fixed, then $s \mapsto A(s, f)$ is continuous.

Example 2. Let f be the holomorphic function given by $f(z) = z$. Then we have

$$A(s, f) = \frac{1}{2\pi} \int_0^{2\pi} \log | \exp(s + i\theta)| d\theta \frac{1}{2\pi} \int_0^{2\pi} \log(\exp(s)) d\theta = s.$$

Example 3. Suppose that $f(z) = \exp(g(z))$, for some holomorphic function g on the annulus $\{z \in \mathbf{C} : a < |z| < b\}$. We then have

$$A(s, f) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}(g(\exp(s + i\theta))) d\theta = \mathrm{Re}(\overline{g}(\exp(s))),$$

where $\overline{g}$ is the radially symmetric holomorphic function given by

$$\overline{g}(z) = \frac{1}{2\pi} \int_0^{2\pi} g(\exp(i\theta) z) d\theta.$$

Note that $\overline{g}$ is a holomorphic function which is constant on each circle $\{z \in \mathbf{C} : |z| = \exp(s)\}$ and is therefore constant. It follows that the function $s \mapsto A(s, f)$ is constant on the interval $(\log(a), \log(b))$.

Example 4. Suppose that the holomorphic function $f(z)$ has no zeroes in the the annulus $\{z \in \mathbf{C} : a < |z| < b\}$. Then f determines a continuous map

$$\{z \in \mathbf{C} : a < |z| < b\} \to \mathbf{C} \setminus \{0\}$$

which has a well-defined *winding number* $N \in \mathbf{Z}$ around the origin. This winding number vanishes if and only if f can be written globally as the exponential of another holomorphic function $g(z)$. We can always arrange this by multiplying f by a suitable power of z: that is, we can write $f(z) = z^N \exp(g(z))$ for some holomorphic function g on $\{z \in \mathbf{C} : a < |z| < b\}$. In this case, we have

$$A(s, f) = A(s, z^N) + A(s, \exp(g)) = Ns + c$$

for some real constant c, by virtue of Examples 2 and 3.

Let us now return to the case of a general holomorphic function $f : \{z \in \mathbf{C} : a < |z| < b\} \to \mathbf{C}$. It follows from Example 4 that the function $s \mapsto A(s, f)$ is piecewise linear with integer slopes: it is linear when restricted to any interval $I \subseteq (\log(a), \log(b))$ whose interior does not contain any point of the form $\log |z|$, where z is a zero of f.

The function $s \mapsto A(s, f)$ fails to be linear exactly when s has the form $\log |z|$, where z is a zero of f. It follows from the above reasoning that the difference $\partial_+ A(s, f) - \partial_- A(s, f)$ is given by the difference between the winding numbers of the functions

$$\theta \mapsto f(\exp(s + \epsilon + i\theta)) \qquad \theta \mapsto f(\exp(s - \epsilon + i\theta))$$

where ϵ is some small real number. This difference is equal to the number of zeros of f on the circle $\{z \in \mathbf{C} : |z| = \exp(s)\}$, counted with multiplicity. In particular, it is a nonnegative integer. Non-negativity implies that the piecewise-linear function $s \mapsto A(s, f)$ is actually convex.

For nonzero $g \in B_{[a,b]}$, the function $s \mapsto v_s(g)$ appearing in Theorem 6 can be regarded as an analogue of the function $s \mapsto A(s, f)$ in the setting of p-adic geometry. More accurately, it is an analogue of the concave function $s \mapsto -A(-s, f)$, where the sign is a matter of convention.

Example 5 (Jensen's Formula)**.** Let f be a holomorphic function defined on the open disk $\{z \in \mathbf{C} : |z| < b\}$, and assume for simplicity that $f(0) \neq 0$. For $t > 0$, set $h(t) = A(\log(t), f)$. Then, for sufficiently small t, the function $h(t)$ takes the constant value $\log |f(0)|$. We therefore have

$$h(t) = \log |f(0)| + \int_0^t h'(r)dr = \log |f(0)| + \int_0^t \frac{A'(\log(r), f)}{r} dr.$$

Here $A'(\log(r), f)$ is a well-defined integer provided that f does not have any zeroes on the circle $\{z \in \mathbf{C} : |z| = r\}$, given by the number $N(r, f)$ of zeroes of f on the disk $\{z \in \mathbf{C} : |z| < r\}$ (counted with multiplicity). We may rewrite the preceding equality as

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(t \exp(i\theta))| = \log |f(0)| + \int_0^t \frac{N(r, f)}{r} dr,$$

which is an identity known as *Jensen's formula.*

Let us now return to the case of interest to us. Fix a perfectoid field $C^\flat$ of characteristic p and real numbers $0 < a \leq b < 1$, and set $\alpha = -\log(a)$, $\beta = -\log(b)$. Recall that, to every nonzero element $f \in B_{[a,b]}$, we can associate a convex piecewise linear function

$$v_\bullet(f) : [\beta, \alpha] \to \mathbf{R} \qquad s \mapsto v_s(f) = -\log |f|_{\exp(-s)}.$$

Heuristically, one can think of the function $s \mapsto v_\bullet(f)$ as a p-adic analogue of the function $s \mapsto -A(-s, f)$ constructed above (here the insertion of signs is really just a matter of convention). We saw in the previous lecture that this function is piecewise linear with integer slopes, and that it is *concave*. Even better, $v_\bullet(f)$ can be promoted to a *germ* of a piecewise linear function on a neighborhood of the interval $[\beta, \alpha]$: in other words, it has well-defined left and right derivatives

$$\partial_- v_\beta(f) \qquad \partial_+ v_\alpha(f),$$

these are integers, given by $\partial_- v_\beta(f')$ and $\partial_+ v_\alpha(f')$ where $f' \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ is any sufficiently close approximation to f with respect to the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. In this case, the function $s \mapsto v_s(f')$ is defined for all $s \in \mathbf{R}_{>0}$ (and is piecewise linear with integer slopes), so the difference

$$\partial_- v_\beta(f) - \partial_+ v_\alpha(f) = \partial_- v_\beta(f') - \partial_+ v_\alpha(f') \geq 0$$

is a nonnegative integer. Our goal is to show that, like in the complex-analytic world, this integer has a concrete interpretation: when $C^\flat$ is algebraically closed, it is equal to the degree of the divisor $\mathrm{Div}_{[a,b]}(f)$ (that is, the number of points of $y \in Y_{[a,b]}$ where the function f vanishes, each counted with multiplicity $\mathrm{ord}_y(f)$). In the last lecture, we reduced the proof of this to the following assertion:

Theorem 6. *Let f be a nonzero element of $B_{[a,b]}$. Then:*

(1) *If $\partial_- v_\beta(f) = \partial_+ v_\alpha(f)$, then f is invertible in $B_{[a,b]}$.*

(2) *If $C^\flat$ is algebraically closed and $\partial_- v_\beta(f) \neq \partial_+ v_\alpha(f)$, then there is a point $y \in Y_{[a,b]}$ such that $f(y) = 0$.*

Remark 7. We saw in the previous lecture that the converse of (1) and (2) hold.

Warning 8. Let f be as in Theorem 6. If $\partial_- v_\beta(f) = \partial_+ v_\alpha(f)$, then the function $s \mapsto v_s(f)$ is linear on the interval $[\beta, \alpha]$. However, the converse is false in general. Note that if $a < b$, then the concavity of $v_\bullet(f)$ yields inequalities

$$\partial_- v_\beta(f) \geq \partial_+ v_\beta(f) \geq \partial_- v_\alpha(f) \geq \partial_+ v_\alpha(f),$$

and that $v_\bullet(f)$ is linear on the interval $[\beta, \alpha]$ if and only if we have an equality $\partial_+ v_\beta(f) \geq \partial_- v_\alpha(f)$. In the case where $C^\flat$ is algebraically closed, this is equivalent to the requirement that f does not vanish at any point of

$$Y_{(a,b)} = \{y = (K, \iota) \in Y : a < |p|_K < b\}.$$

However, it is possible for f to satisfy this condition while vanishing at points $y = (K, \iota)$ satisfying $|p|_K = a$ or $|p|_K = b$, in which case one of the inequalities

$$\partial_- v_\beta(f) \geq \partial_+ v_\beta(f) \qquad \partial_- v_\alpha(f) \geq \partial_+ v_\alpha(f)$$

would be strict and f would not be invertible.

Let us now prove part (1) of Theorem 6 (we will prove (2) in a future lecture). Assume first that f belongs to the ring $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$, and therefore admits a unique Teichmüller expansion

$$f = \sum_{n \gg -\infty} [c_n] p^n$$

where the real numbers $|c_n|_{C^\flat}$ are bounded. In this case, the function $s \mapsto v_s(f)$ is defined for all $s > 0$, and given by the formula

$$v_s(f) = \inf_{n \in \mathbf{Z}} (v(c_n) + ns).$$

The equality $\partial_- v_\beta(f) = \partial_+ v_\alpha(f)$ is equivalent to the requirement that the function $s \mapsto v_s(f)$ is linear in a small neighborhood of the interval $[\beta, \alpha]$, and therefore coincides with the linear function

$$s \mapsto v(c_{n_0} s) + n_0 s = v_s([c_{n_0}] p^{n_0})$$

on that neighborhood; it follows that we have

$$v(c_n) + ns > v(c_{n_0}) + n_0 s$$

for $n_0 \neq n$ and $s \in [\beta, \alpha]$. Restated in terms of absolute values, we have

$$|c_{n_0}|_{C^\flat} \rho^{n_0} > |c_n|_{C^\flat} \rho^n$$

for all $n \neq n_0$ and $\rho \in [a, b]$.

We wish to show that in this case, the function f is invertible. Replacing f by the quotient $\frac{f}{[c_{n_0}] p^{n_0}}$, we can reduce to the case where $n_0 = 0$ and $[c_{n_0}] = 1$, so that our inequality can be rewritten as

$$1 > |c_n|_{C^\flat} \rho^n$$

for $n \neq 0$ and $\rho \in [a, b]$. Setting $\epsilon = \sum_{n \neq 0} [c_n] p^n$, we have $|\epsilon|_\rho = \sup\{|c_n| \rho^n\}_{n \neq 0} < 1$. It follows that ϵ is topologically nilpotent in the ring $B_{[a,b]}$, so that $f = 1 + \epsilon$ has an inverse given by the convergent sum

$$f^{-1} = (1 + \epsilon)^{-1} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \cdots .$$

This completes the proof of part (1) in the special case where f belongs to $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$.

We now treat the general case. Let f be a nonzero element of $B_{[a,b]}$, which we write as the limit of a sequence

$$f_1, f_2, f_3, \cdots \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$$

which is Cauchy for the Gauss norms $| \bullet |_\rho$ for $\rho \in [a, b]$. We proved in Lecture 11 that we have

$$\partial_- v_\beta(f) = \partial_- v_\beta(f_n) \qquad \partial_+ v_\alpha(f) = \partial_+ v_\alpha(f_n)$$

for $n \gg 0$. Passing to a subsequence, we may assume that these equalities hold for all n. In this case, our hypothesis $\partial_- v_\beta(f) = \partial_+ v_\alpha(f)$ guarantees that we also ahve $\partial_- v_\beta(f_n) = \partial_+ v_\alpha(f_n)$ for each n. By the special case treated above, this means that each f_n is invertible when regarded as an element of the completion $B_{[a,b]}$. Let us denote the inverse by f_n^{-1}. For each $\rho \in [a, b]$, we have

$$|f_m^{-1} - f_n^{-1}|_\rho = |\frac{f_n - f_m}{f_m \cdot f_n}|_\rho = \frac{|f_n - f_m|_\rho}{|f_m|_\rho \cdot |f_n|_\rho}.$$

As m and n tend to infinity, the numerator of this expression goes to zero (since the sequence $\{f_n\}$ is Cauchy with respect to the Gauss norm $| \bullet |_\rho$) and the denominator tends to $|f|_\rho^2$. It follows that the quantity $|f_m^{-1} - f_n^{-1}|_\rho$ tends to zero. That is, $\{f_n^{-1}\}$ is a Cauchy sequence with respect to each of the Gauss norms $| \bullet |_\rho$ for $\rho \in [a, b]$, and therefore converges to an element $g \in B_{[a,b]}$. It follows by continuity that $f \cdot g = 1$, so that f is invertible as desired.