Lecture 12: Detection of Zeroes

October 29, 2018

Throughout this lecture, we fix an algebraically closed perfectoid field $C^\flat$ of characteristic p. Then every untilt K of $C^\flat$ is algebraically closed, so the map $\sharp : C^\flat \to K$ is surjective (we will prove this later).

Notation 1. We let Y denote the set of isomorphism classes of untilts (K, ι) of $C^\flat$. We will use the letter y to denote a typical point of Y. For $0 < a \leq b < 1$, we let $Y_{[a,b]}$ denote the subset of Y consisting of those points $y = (K, \iota)$ satisfying $a \leq |p|_K \leq b$.

We use letters like f and g to denote typical elements of the ring $B_{[a,b]}$. For $y \in Y_{[a,b]}$, we let $f(y)$ denote the image of f under the ring homomorphism $B_{[a,b]} \to K$ constructed in Lecture 5. We also let $\mathrm{ord}_y(f)$ denote the order of vanishing of f at the point y (so that $\mathrm{ord}_y(f) > 0$ if and only if $f(y) = 0$).

Our goal, over the next few lectures, is to prove the following result which was promised in Lecture 10:

Theorem 2. *Let f be a nonzero element of $B_{[a,b]}$. Then:*

(1) *The divisor $\mathrm{Div}_{[a,b]}(f) = \sum_{y \in B_{[a,b]}} \mathrm{ord}_y(f) \cdot y$ is finite. In other words, the order of vanishing $\mathrm{ord}_y(f)$ is finite for all points $y \in B_{[a,b]}$, and vanishes for all but finitely many points of $B_{[a,b]}$.*

(2) *Let g be another element of $B_{[a,b]}$. If $\mathrm{Div}_{[a,b]}(g) \geq \mathrm{Div}_{[a,b]}(f)$ (that is, if $\mathrm{ord}_y(g) \geq \mathrm{ord}_y(f)$ for all $y \in B_{[a,b]}$), then g is (uniquely) divisible by f.*

Let us first dispense with the uniqueness.

Proposition 3. *The ring $B_{[a,b]}$ is an integral domain.*

Proof. Set $\alpha = -\log(a)$ and $\beta = -\log(b)$. For each $f \in B_{[a,b]}$, we have a function

$$[\beta, \alpha] \to \mathbf{R} \cup \{\infty\} \qquad s \mapsto v_s(f) = -\log |f|_{\exp(-s)}.$$

We saw in the previous lecture that if f is not zero, then $v_\bullet(f)$ is a piecewise linear concave function on the interval $[\beta, \alpha]$ (with integer slopes): in particular, it is everywhere finite. If g is also nonzero, then $v_\bullet(g)$ is also a piecewise linear concave function. It follows that the function $v_s(fg) = v_s(f) + v_s(g) < \infty$ for all $s \in [\beta, \alpha]$, so that fg is nonzero in $B_{[a,b]}$. □

We now consider a special case of Theorem 2, where $f = \xi$ is a *distinguished* element of the ring $\mathbf{A}_{\mathrm{inf}}$. In this case, assertion (1) is clear and (2) reduces to the following:

Proposition 4. *Let ξ be a distinguished element of $\mathbf{A}_{\mathrm{inf}}$ which vanishes at a point $y \in Y_{[a,b]}$. If $g \in B_{[a,b]}$ also vanishes at y, then g is divisible by ξ (uniquely, by virtue of Proposition 3).*

Proof. Suppose first that g belongs to the ring $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. Then we can write $g = \frac{g_0}{p^m [\pi]^n}$ for some $g_0 \in \mathbf{A}_{\mathrm{inf}}$ and some $m, n \geq 0$. Since $g(y) = 0$, it follows that $g_0(y) = 0$: that is, g_0 belongs to the kernel of the map

$\theta : \mathbf{A}_{\mathrm{inf}} \to \mathcal{O}_K$ determined by the untilt $y = (K, \iota)$. We saw in Lecture 3 that this kernel is generated by the distinguished element ξ. We can therefore write $g_0 = \xi \cdot h_0$ for some $h_0 \in \mathbf{A}_{\mathrm{inf}}$. It follows that $g = \xi \cdot h$, where $h = \frac{h_0}{p^m [\pi]^n}$.

We now treat the general case. Let g be any element of $B_{[a,b]}$. Then we can write g as the limit of a sequence $g_1, g_2, g_3, \ldots \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. It follows that $g(y)$ is the limit of the sequence $\{g_i(y)\}$ in the field K corresponding to the point $y = (K, \iota) \in Y_{[a,b]}$. For each $i > 0$, write $g_i(y) = c_i^{\sharp}$ for some element $c_i \in C^{\flat}$. If $g(y) = 0$, then the sequence $\{g_i(y)\}$ converges to zero in K. Since $|g_i(y)|_K = |c_i^{\sharp}|_K = |c_i|_{C^{\flat}}$, it follows that the sequence $\{c_i\}$ converges to zero in $C_{\flat}$, and therefore the sequence of Teichmüller representatives $\{[c_i]\}$ converges to zero with respect to the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. It follows that the sequence $\{g_i - [c_i]\}$ also converges to g. Replacing each g_i by $g_i - [c_i]$, we can assume that each g_i vanishes at the point y. By the first part of the proof, we can write $g_i = \xi \cdot h_i$, for some (uniquely determined) $h_i \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]$. We will complete the proof by showing that $\{h_i\}$ is a Cauchy sequence (for the Gauss norms $|\bullet|_a$ and $|\bullet|_b$): then it converges to a unique element $h \in B_{[a,b]}$ which satisfies $g = \xi \cdot h$ by continuity. To prove this, we observe that

$$|g_i - g_j|_a = |\xi \cdot (h_i - h_j)|_a = |\xi|_a \cdot |h_i - h_j|_a$$

so that

$$|h_i - h_j|_a = \frac{|g_i - g_j|_a}{|\xi|_a} \to 0$$

as $i, j \to \infty$ (and similarly for the Gauss norm $|\bullet|_b$). □

Proposition 4 suggests a strategy for proving Theorem 2. Let f be a nonzero element of $B_{[a,b]}$, and let g be another element satisfying $\mathrm{Div}_{[a,b]}(g) \geq \mathrm{Div}_{[a,b]}(f)$. Suppose that $\mathrm{Div}_{[a,b]}(f)$ is not zero: that is, there is some point $y_1 \in Y_{[a,b]}$ such that $f(y_1) = 0$. Then we also have $g(y_1) = 0$. Let $\xi_1 \in \mathbf{A}_{\mathrm{inf}}$ be a distinguished element which vanishes at y_1. Applying Proposition 4, we can write $f = \xi_1 \cdot f_1$ and $g = \xi_1 \cdot g_1$ for some elements $f_1, g_1 \in B_{[a,b]}$. Then we have $\mathrm{Div}_{[a,b]}(g_1) \geq \mathrm{Div}_{[a,b]}(f_1)$. We can then repeat the argument: if $\mathrm{Div}_{[a,b]}(f_1) \neq 0$, we can find another distinguished element ξ_2 vanishing at a point $y_2 \in Y_{[a,b]}$ such that $f_1 = \xi_2 \cdot f_2$ and $g_1 = \xi_2 \cdot g_2$. Continuing in this way, we obtain sequences $\{f_n\}$, $\{g_n\}$, and $\{\xi_n\}$ satisfying

$$f = \xi_1 \cdot \xi_2 \cdots \xi_n \cdot f_n \qquad g = \xi_1 \cdot \xi_2 \cdots \xi_n \cdot g_n.$$

To prove Theorem 2, we must show the following:

- (1′) This process eventually stops: that is, we eventually end up in a situation where $\mathrm{Div}_{[a,b]}(f_n) = 0$. In this case, we have $\mathrm{Div}_{[a,b]}(f) = y_1 + y_2 + \cdots + y_n$.
- (2′) When the process stops, the element f_n is a unit (and therefore automatically divides g_n).

We can prove (1′) very easily from the results of the previous lecture. Let h be any nonzero element of the ring $B_{[a,b]}$, and write h as the limit of a sequence $\{h_i \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\pi]}]\}$ which converges for the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. In the previous lecture, we proved that the sequence of functions $\{v_{\bullet}(h_i)\}$ is eventually constant on the interval $[\beta, \alpha]$. Moreover, we can do a little better: the sequence of left derivatives $\partial_- v_\beta(h_i)$ and right derivatives $\partial_+ v_\alpha(h_i)$ are eventually constant, converging to integers $\partial_- v_\beta(h)$ and $\partial_+ v_\alpha(h)$, respectively.

Exercise 5. Check that the integers $\partial_- v_\beta(h)$ and $\partial_+ v_\alpha(h)$ depend only on h, and not on the choice of Cauchy sequence $\{h_i\}$ converging to h.

Since each of the functions $v_{\bullet}(h_i)$ is concave, we have $\partial_- v_\beta(h_i) \geq \partial_+ v_\alpha(h_i)$ for each i, and therefore $\partial_- v_\beta(h) \geq \partial_+ v_\beta(h)$.

Proposition 6. *Let f be a nonzero element of $B_{[a,b]}$, and set $N = \partial_- v_\beta(f) - \partial_+ v_\alpha(f) \geq 0$. Then the construction sketched above must terminative in $\leq N$ steps. That is, f cannot be divisible by a product $\xi_1 \cdot \xi_2 \cdot \cdots \cdot \xi_{N+1}$ of distinguished elements ξ_i vanishing at points $y_i \in Y_{[a,b]}$.*

Proof. We saw in the previous lecture that if ξ is a distinguished element of $\mathbf{A}_{\text{inf}}$, then the function $s \mapsto v_s(\xi)$ is given by the formula

$$v_s(\xi) = \begin{cases} s & \text{if } s \leq v(\xi) \\ v(\xi) & \text{otherwise.} \end{cases}.$$

In particular, if the point at which ξ vanishes belongs to $Y_{[a,b]}$, then $v(\xi)$ belongs to the interval $[\beta, \alpha]$, and therefore

$$\partial_- v_\beta(\xi) = 1 \qquad \partial_+ v_\alpha(\xi) = 0.$$

In particular, if we can write $f = \xi_1 \cdot \xi_2 \cdots \xi_{N+1} \cdot f_{N+1}$, then we have

$$\begin{aligned} N &= \partial_- v_\beta(f) - \partial_+ v_\alpha(f) \\ &= (\sum_{i=1}^{N+1} \partial_- v_\beta(\xi_i) - \partial_+ v_\alpha(\xi_i)) + (\partial_- v_\beta(f_{N+1}) - \partial_+ v_\alpha(f_{N+1}) \\ &\geq (\sum_{i=1}^{N+1} \partial_- v_\beta(\xi_i) - \partial_+ v_\alpha(\xi_i)) \\ &= N+1 \end{aligned}$$

which is a contradiction. □

We will deduce Theorem 2 from the following:

Theorem 7. *Let f be a nonzero element of $B_{[a,b]}$. The following conditions are equivalent:*

- (a) *The element f is invertible in $B_{[a,b]}$.*
- (b) *The integer $N = \partial_- v_\beta(f) - \partial_+ v_\alpha(f)$ is equal to zero (in particular, the concave function $s \mapsto v_s(f)$ is linear on the interval $[\beta, \alpha]$).*
- (c) *The divisor $\mathrm{Div}_{[a,b]}(f)$ is equal to zero. That is, there is no point $y \in Y_{[a,b]}$ such that $f(y) = 0$.*

Note that the implication $(b) \Rightarrow (c)$ follows from Proposition 6. The implication $(a) \Rightarrow (b)$ is also clear: if f has an inverse $f^{-1} \in B_{[a,b]}$, then it is easy to see that

$$\partial_- v_\beta(f^{-1}) = -\partial_- v_\beta(f) \qquad \partial_+ v_\alpha(f^{-1}) = -\partial_+ v_\alpha(f)$$

so that

$$N = \partial_- v_\beta(f) - \partial_+ v_\alpha(f) = -(\partial_- v_\beta(f^{-1}) - \partial_+ v_\alpha(f^{-1}) \leq 0.$$

Over the next few lectures, we will show (using different arguments) that both of the implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ is reversible. Once that is done, it will follow that $(c) \Rightarrow (a)$: that is, a nonzero element $f \in B_{[a,b]}$ satisfying $\mathrm{Div}_{[a,b]}(f) = 0$ is invertible. This will complete the proof of $(2')$ and with it the proof of Theorem 2.