Lecture 11: Trivial Eigenspaces of the Frobenius

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Throughout this lecture, we fix a perfectoid field $C^\flat$ of characteristic p. Our goal is to prove the following result, which was stated without proof in the previous lecture:

Theorem 1. *Let n be a negative integer. Then $B^{\varphi = p^n} = \{0\}$.*

Recall that Theorem 1 is consistent with the analysis of Lecture 6: a bi-infinite Teichmüller expansion which "obviously" belongs to the eigenspace $B^{\varphi=p^n}$ must be identically zero. For example, when $n = -1$, such a Teichmüller expansion would look like

$$\sum_{m \in \mathbf{Z}} p^m [c]^{p^m}$$

for some $c \in C^\flat$. Such a sum cannot converge for *any* of the Gauss norms $| \bullet |_\rho$ unless c is equal to zero.

This does not translate directly to a proof of Theorem 1, because we do not have existence and uniqueness of Teichmüller expansions for elements of the ring B. However, it suggests an approach to the problem: assuming the existence of a nonzero element $f \in B^{\varphi=p^n}$, we might hope to derive a contradiction (when n is negative) by studying the properties of the Gauss norms $|f|_\rho$. Here it will be useful not to focus on a particular value of ρ, but instead to study the function $\rho \mapsto |f|_\rho$ (where the element f is fixed). As we will see in a moment, the properties of this function are more apparent if we make a logarithmic change of variable.

Notation 2. Let K be a field equipped with a non-archimedean absolute value $| \bullet |_K$. For each element $x \in K$, we set $v(x) = -\log |x|_K$, which we regard as an element of $\mathbf{R} \cup \{\infty\}$. We refer to $v(x)$ as the *valuation of x*. Note that we have

$$\mathcal{O}_K = \{x \in K : v(x) \geq 0\} \qquad \mathfrak{m}_K = \{x \in K : v(x) > 0\} \qquad (v(x) = \infty) \Leftrightarrow (x = 0).$$

$$v(xy) = v(x) + v(y) \qquad v(x+y) \geq \min(v(x), v(y)).$$

Remark 3. Recall that, if K is a field equipped with a non-archimedean absolute value $| \bullet |_K$, then for any $\alpha > 0$, the absolute value $| \bullet |_K^\alpha$ defines the same topology on K. In other words, for many purposes, it is useful to regard the absolute value on K as only well-defined up to a constant exponent. Likewise, it is useful to regard the valuation v on K as only well-defined up to a constant factor. In the case of a discrete valuation, it is often convenient to normalize so that the map $v : K \to \mathbf{R} \cup \{\infty\}$ takes values in $\mathbf{Z} \cup \{\infty\}$. However, we are interested in perfectoid fields where such a normalization is impossible.

Notation 4. Let f be an element of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. Then, for every positive real number s, we define $v_s(f) \in \mathbf{R} \cup \{\infty\}$ by the formula

$$v_s(f) = -\log |f|_{\exp(-s)}.$$

More concretely, if f has a Teichmüller expansion $\sum_{n \gg -\infty} [c_n] p^n$, we have

$$|f|_\rho = \sup\{|c_n|_{C^\flat} \rho^n\} \qquad v_s(f) = \inf\{v(c_n) + ns\}.$$

In particular, we have $v_s(f) = \infty$ if and only if $f = 0$.

Note that, from the multiplicativity and ultrametric properties of the Gauss norms, we have

$$v_s(fg) = v_s(f) + v_s(g) \qquad v_s(f+g) \geq \min(v_s(f), v_s(g)).$$

Suppose that f is nonzero. Note that for a fixed positive real number s, the infimum $v_s(f) = \inf\{v(c_n) + ns\}$ is achieved for finitely many values of n, taken from some set $\{n_0 < n_1 < \cdots < n_k\}$. It follows that, for ϵ sufficiently small, we have

$$v_{s+\epsilon}(f) = v_s(f) + n_0\epsilon \qquad v_{s-\epsilon}(f) = v_s(f) - n_k\epsilon.$$

This proves the following:

Proposition 5. *Let f be a nonzero element of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. Then the construction $s \mapsto v_s(f)$ determines a function*

$$v_{\bullet}(f) : \mathbf{R}_{>0} \to \mathbf{R}$$

which is piecewise linear with integer slopes. Moreover, it is concave (that is, the slopes are decreasing).

Example 6. Let f be a distinguished element of $\mathbf{A}_{\inf}$, so that f admits a Teichmüller expansion $\sum_{n \geq 0}[c_n]p^n$ where $|c_0|_{C^\flat} < 1$ and $|c_1|_{C^\flat} = 1$. We then have

$$v_s(f) = \min(v(c_0) + 0 \cdot s, v(c_1) = 1 \cdot s) = \min(v(c_0), s).$$

Remark 7. Let f be a nonzero element of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. It follows from Proposition 5 that the function $v_{\bullet}(f)$ has well-defined left and right derivatives at each point $s \in \mathbf{R}_{>0}$ (though the left and right derivatives need not be the same). We will denote the values of these left and right derivatives (at a point $s \in \mathbf{R}_{>0}$) by $\partial_- v_s(f)$ and $\partial_+ v_s(f)$, respectively. Note that $\partial_- v_s(f)$ and $\partial_+ v_s(f)$ are integers satisfying $\partial_- v_s(f) \geq \partial_+ v_s(f)$.

Our next goal is to extend the definition of $v_s(f)$ to the case where f belongs to a suitable completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$. Note that if we are given a sequence of elements $f_1, f_2, \ldots \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ which converges for the Gauss norm $|\bullet|_{\exp(-s)}$, then the sequence of Gauss norms $|f_i|_{\exp(-s)}$ must converge. In fact, we can do a bit better:

Proposition 8. *Let s be a positive real number and let $f_1, f_2, \ldots \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ be a Cauchy sequence for the Gauss norm $|\bullet|_{\exp(-s)}$ which does not converge to zero. Then the sequences*

$$v_s(f_i) \qquad \partial_- v_s(f_i) \qquad \partial_+ v_s(f_i)$$

are eventually constant.

Proof. Set $a = \lim_{i \to \infty} v_s(f_i) = -\log(\lim_{i \to \infty} |f_i|_{\exp(-s)})$. We can then choose an integer $n \gg 0$ such that, for $n' > n$, we have $v_s(f_{n'} - f_n) > a$. Then $v_s(f_n) = a$. It follows by continuity that, for any fixed $n' > n$, there exists a small interval $I = (s - \epsilon, s + \epsilon)$ (depending on n') such that $v_t(f_{n'} - f_n) > v_t(f_n)$ for $t \in I$. It then follows that $v_t(f_{n'}) = v_t(f_n)$ for $t \in I$, so that

$$v_s(f_{n'}) = v_s(f_n) \qquad \partial_- v_s(f_{n'}) = \partial_- v_s(f_n) \qquad \partial_+ v_s(f_{n'}) = \partial_+ v_s(f_n).$$

□

Corollary 9. *Fix $0 < a \leq b < 1$. Let $f_1, f_2, \ldots \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ be a Cauchy sequence for the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. Assume that the sequence $\{f_n\}$ does not converge to zero for both of the Gauss norms $|\bullet|_a$ and $|\bullet|_b$. Then the sequence of functions $\{s \mapsto v_s(f_n)\}$ is eventually constant on the closed interval $[-\log(b), -\log(a)]$.*

Proof. Without loss of generality we may assume that each f_n is nonzero. We also assume that the sequence $\{f_n\}$ does not converge to zero with respect to $|\bullet|_b$ (the proof in the other case is similar. Set $s = -\log(b)$. It follows from Proposition 8 that we can choose an integer $m \gg 0$ such that the sequence of real numbers $\{v_s(f_{m'})\}_{m' \geq m}$ takes some constant value $r \in \mathbf{R}$ and the sequence of integers $\{\partial_+ v_s(f_{m'})\}_{m' \geq m}$ takes some constant value $k \in \mathbf{Z}$. Since each of the functions $v_\bullet(f_{m'})$ is concave, it follows that $v_t(f_{m'}) \leq r + k(t-s)$ for $t \geq s$. In particular, each $v_t(f_{m'})$ is bounded above by $r' = \max(r, r + k\log(\frac{b}{a}))$ on the interval $[-\log(b), -\log(a)]$.

Choose $n \gg m$ such that, for $n' \geq n$, we have $|f_{n'} - f_n|_\rho < \exp(-r')$ for $\rho \in [a, b]$, or equivalently we have

$$v_t(f_{n'} - f_n) > r'$$

for $t \in [-\log(b), -\log(a)]$. Combining this with the inequality $v_t(f_n) \leq r'$, we conclude that $v_t(f_n) = v_t(f_{n'})$ for $t \in [-\log(b), -\log(a)]$. □

Note that, for $f \in B_{[a,b]}$, the Gauss norm $|f|_\rho$ is well-defined for $\rho \in [a, b]$. Consequently, we can define $v_s(f) = -\log |f|_{\exp(-s)}$ for $s \in [-\log(b), -\log(a)]$. Write f as the limit of a sequence $\{f_n\}$ in $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\pi]}]$ which converges for the norms $|\bullet|_a$ and $|\bullet|_b$. If $f \neq 0$, then the sequence $\{f_n\}$ cannot converge to zero with respect to both $|\bullet|_a$ and $|\bullet|_b$. Applying Corollary 9, we deduce that there exists an integer $n \gg 0$ such that $v_s(f) = v_s(f_n)$ for each $s \in [-\log(b), -\log(a)]$. Combining this observation with Proposition 5, we obtain the following:

Corollary 10. *Let f be a nonzero element of $B_{[a,b]}$. Then the construction $s \mapsto v_s(f)$ determines a function*

$$v_\bullet(f) : [-\log(b), -\log(a)] \to \mathbf{R}$$

which is piecewise linear with integer slopes. Moreover, it is concave (that is, the slopes are decreasing).

Remark 11 (The Hadamard Three-Circle Theorem)**.** Corollary 10 has a counterpart in complex analysis. Suppose we are given positive real numbers $a < b$, and let $f : \{z \in \mathbf{C} : a \leq |z| \leq b\} \to \mathbf{C}$ be a continuous function which is holomorphic on the interior of the annulus. Then the classical *Hadamard three-circle theorem* asserts that the function

$$s \mapsto \sup_{|z| = e^s} |f(z)|$$

is convex on the interval $[\log(a), \log(b)]$ (for closer analogue in the setting of complex analysis, see Lecture 13).

Corollary 12. *Let f be a nonzero element of B. Then the construction $s \mapsto v_s(f)$ determines a function*

$$v_\bullet(f) : \mathbf{R}_{>0} \to \mathbf{R}$$

which is piecewise linear with integer slopes. Moreover, it is concave (that is, the slopes are decreasing).

We now turn to the proof of Theorem 1. First, we recall that the Gauss norms satisfy the identities

$$|\varphi(f)|_{\rho^p} = |f|_\rho^p \qquad |p^n f|_\rho = \rho^n |f|_\rho.$$

We can rewrite these identities as

$$v_{ps}(\varphi(f)) = p v_s(f) \qquad v_s(p^n f) = ns + v_s(f).$$

Consequently, if f is a nonzero element of B satisfying $\varphi(f) = p^n f$, we have

$$p v_{s/p}(f) = v_s(\varphi(f)) = v_s(p^n f) = ns + v_s(f).$$

For $s > 0$, set $h(s) = \partial_+ v_s(f)$. Right-differentiating the preceding identity with respect to s, we obtain an equality

$$h(s/p) = n + h(s).$$

However, since $s \mapsto v_s(f)$ is concave, the function h must be nonincreasing. We therefore have

$$h(s) \leq h(s/p) = n + h(s)$$

so that $n \geq 0$.