Lecture 10: Structure of the Fargues-Fontaine Curve

October 29, 2018

Throughout this lecture, we fix an *algebraically closed* perfectoid field $C^\flat$ of characteristic p, with valuation ring $\mathcal{O}_C^\flat$. Let Y denote the set of all isomorphism classes of characteristic zero untilts $y = (K, \iota)$ of $C^\flat$. To each nonzero element f of the ring B, we associate the "divisor"

$$\sum_{y \in Y} \mathrm{ord}_y(f) \cdot y$$

(which is generally an infinite sum, though "locally" finite). We recall three results from the previous lecture (which we have not yet proved):

Theorem 1. (1) *Every nonzero element $f \in B$, has finite order of vanishing $\mathrm{ord}_y(f)$ at each point $y \in Y$.*

(2) *Another nonzero element $g \in B$ is divisible by f if and only if $\mathrm{Div}(f) \leq \mathrm{Div}(g)$: that is, $\mathrm{ord}_y(f) \leq \mathrm{ord}_y(g)$ for each $y \in Y$.*

Theorem 2. *For $n < 0$, the eigenspace $B^{\varphi = p^n}$ vanishes.*

Theorem 3. *Every untilt of $C^\flat$ is algebraically closed.*

Let us now collect some consequences.

Corollary 4. *The ring $B^{\varphi=1}$ is a field.*

In fact, the field $B^{\varphi=1}$ can be identified with $\mathbf{Q}_p$; we stated this without proof in the previous lecture, but will not need it yet.

Proof of Corollary 4. Let f be a nonzero element of $B^{\varphi=1}$; we wish to prove that f is invertible in B (in which case it is clear that the inverse f^{-1} also belongs to $B^{\varphi=1}$. By virtue of Theorem 1, it will suffice to show that the divisor $\mathrm{Div}(f)$ vanishes. Since f is fixed by the Frobenius, the divisor $\mathrm{Div}(f)$ is likewise fixed by the Frobenius. Consequently, if $\mathrm{Div}(f) \neq 0$, then we can write $\mathrm{Div}(f) \geq \sum_{n \in \mathbf{Z}} \varphi^n(y)$ for some $y = (K, \iota) \in Y$. It follows from Theorem 3 that K contains a copy of $\mathbf{Q}_p^{\mathrm{cyc}}$, so that we can write $\sum_{n \in \mathbf{Z}} \varphi^n(y) = \log([\epsilon])$ for some $\epsilon \in 1 + \mathfrak{m}_C^\flat$. Applying Theorem 1, we can write $f = g \cdot \log([\epsilon])$. It follows that $g \in B^{\varphi = p^{-1}} = \{0\}$, contradicting our assumption that $f \neq 0$. □

Corollary 5. *For $n \geq 0$, every nonzero element $f \in B^{\varphi=p^n}$ factors as a product $\lambda \log([\epsilon_1]) \cdots \log([\epsilon_n])$ for some $\lambda \in B^{\varphi=1}$ and $\epsilon_1, \ldots, \epsilon_n \in 1 + \mathfrak{m}_{C^\flat}$. Moreover, the factors are uniquely determined up reordering and multiplication by elements of $\mathbf{Q}_p^\times$.*

Proof. We prove existence by induction on n. If $n = 0$, there is nothing to prove. we will therefore assume that $n > 0$. Note that if $\mathrm{Div}(f) = 0$, then f is invertible (Theorem 1) and the inverse f^{-1} belongs to $B^{\varphi = p^{-n}}$, contradicting Theorem 2. As in the proof of Corollary 4, we learn that f is divisible by $\log([\epsilon])$ for some element $\epsilon \neq 1$ of $1+\mathfrak{m}_C^\flat$. Writing $f = g \cdot \log([\epsilon])$, we conclude that $g \in B^{\varphi = p^{n-1}}$. It follows from our inductive hypothesis that we can write $g = \lambda \log([\epsilon_1]) \cdots \log([\epsilon_{n-1}])$ for some $\lambda \in B^{\varphi=1}$ and $\epsilon_1, \ldots, \epsilon_n \in 1 + \mathfrak{m}_{C^\flat}$, so that $f = \lambda \log([\epsilon_1]) \cdots \log([\epsilon_{n-1}]) \cdot \log([\epsilon])$.

To prove uniqueness, it will suffice to show for $1 \neq \epsilon \in 1 + \mathfrak{m}_C^\flat$, the element $\log([\epsilon])$ is a *prime* element of the graded ring $\bigoplus_{n \geq 0} B^{\varphi = p^n}$: that is, if $\log([\epsilon])$ divides a product $f \cdot g$, then either $\log([\epsilon])$ divides f or $\log([\epsilon])$ divides g. Since $\log([\epsilon])$ is homogeneous, it suffices to check this in the case where f and g are homogeneous: that is, we may assume that $f \in B^{\varphi = p^m}$ and $g \in B^{\varphi = p^n}$. Choose a point $y \in Y$ belonging to the vanishing locus of $\log([\epsilon])$. Then either f or g must vanish at the point y; without loss of generality, we may assume that $f(y) = 0$. The equation $\varphi(f) = p^m f$ guarantees that the divisor $\mathrm{Div}(f)$ is Frobenius-invariant, so we must have $\mathrm{Div}(f) \geq \sum_{n \in \mathbf{Z}} \varphi^n(y) = \mathrm{Div}(\log([\epsilon]))$. Applying Theorem 1, we conclude that $\log([\epsilon])$ divides f. $\square$

Let P denote the graded ring $\bigoplus_{n \geq 0} B^{\varphi = p^n}$. Recall that the *Fargues-Fontaine curve* X_{FF} is defined to be the scheme $\mathrm{Proj}(P)$. By definition, the points of X_{FF} (as a topological space) can be identified with homogeneous prime ideals $\mathfrak{p} \subseteq P$ which do not contain the "irrelevant" ideal $\bigoplus_{n > 1} B^{\varphi = p^n}$. Let us give two examples of such ideals:

- It follows from Theorem 1 that B is an integral domain. Consequently, the graded ring P is also an integral domain, so the zero ideal $(0) \subseteq P$ is prime. This prime ideal corresponds to the *generic point* of the Fargues-Fontaine curve X_{FF}.
- Let $(K, \iota) \in Y$ be a characteristic zero untilt of $C^\flat$, and choose an element $\epsilon \in 1 + \mathfrak{m}_C^\flat$ such that $\epsilon \neq 1$ and $\log([\epsilon])$ vanishes at K. It follows from the proof Corollary 5 that the principal ideal $(\log([\epsilon]))$ is prime, and therefore corresponds to a point of the Fargues-Fontaine curve that we will denote by x_K. Note that multiplying $\log([\epsilon])$ by a unit in $\mathbf{Q}_p$ does not change the principal ideal $(\log([\epsilon]))$. Consequently, the point x_K depends only on the untilt K. Moreover, we have $x_K = x_{K'}$ if and only if K and K' belong to the same Frobenius orbit of Y.

We now show that these are the *only* points of the Fargues-Fontaine curve:

Proposition 6. *Let x be a point of the Fargues-Fontaine curve X_{FF} which is not the generic point. Then we have $x = x_K$ for some point $(K, \iota) \in Y$. Moreover, the residue field of X_{FF} at the point x can be identified with K.*

Proof. By construction, the scheme $X_{\mathrm{FF}} = \mathrm{Proj}(P)$ can be obtained by gluing together open affine subschemes of the form $P[\frac{1}{f}]^0 = B[\frac{1}{f}]^{\varphi=1}$, where f is a nonzero homogeneous element of P having positive degree. Let us suppose that x belongs to one of these open subschemes, and therefore corresponds to a nonzero prime ideal $\mathfrak{p} \subseteq B[\frac{1}{f}]^{\varphi=1}$. Choose an element of $\mathfrak{p}$ and write it as a fraction $\frac{g}{f^n}$ for some element $g \in B^{\varphi = p^n}$. It follows from Corollary 5 that, after scaling by a unit, we may assume that this element factors as a product $\frac{\log([\epsilon_1])}{f} \cdots \frac{\log([\epsilon_n])}{f}$. Since $\mathfrak{p}$ is prime, we may assume that it contains one of the factors, which we write as $\frac{\log([\epsilon])}{f}$. Let $y = (K, \iota) \in Y$ be a point at which $\log([\epsilon])$ vanishes. We claim that $x = x_K$, or equivalently that $\mathfrak{p}$ is generated by $\frac{\log([\epsilon])}{f}$. To prove this (and the last claim of Proposition 6), it will suffice to show that the principal ideal $(\frac{\log([\epsilon])}{f})$ is maximal, and that the quotient field

$$B[\frac{1}{f}]^{\varphi=1}/(\frac{\log([\epsilon])}{f})$$

can be identified with K. Since f does not vanish at K, we have a canonical ring homomorphism

$$\rho : B[\frac{1}{f}]^{\varphi=1} \subseteq B[\frac{1}{f}] \to K;$$

we claim that ρ is a surjection whose kernel is generated by $\frac{\log([\epsilon])}{f}$

To prove surjectivity, we note that ρ is already surjective when restricted to $\frac{1}{f}B^{\varphi=p}$, since every element of K has the form $\log(y^\sharp)$ for some $y \in 1 + \mathfrak{m}_C^\flat$ (see Lecture 9). To prove injectivity, we can use Corollary 5 to write every element of $B[\frac{1}{f}]^{\varphi=1}$ as a product $\lambda\frac{\log([\epsilon_1])}{f}\cdots\frac{\log([\epsilon_n])}{f}$. If this point belongs to $\ker(\rho)$, then some fraction $\frac{\log([\epsilon_i])}{f}$ must be annihilated by ρ. The desired result then follows from the observation that $\frac{\log([\epsilon_i])}{f}$ and $\frac{\log([\epsilon])}{f}$ differ by multiplication by some nonzero element of $\mathbf{Q}_p$. □

Corollary 7. *The construction $y = (K, \iota) \mapsto x_K$ induces a bijection*

$$Y/\varphi^{\mathbf{Z}} \simeq \{\textit{Closed points of } X_{\mathrm{FF}}\}$$

Corollary 8. *The Fargues-Fontaine curve X_{FF} is a Dedekind scheme.*

Proof. By definition, we can cover X_{FF} by open affine subschemes of the form $R = B[\frac{1}{f}]^{\varphi=1}$. The proof of Proposition 6 shows that every nonzero prime ideal of R is a maximal ideal generated by a single element. In particular, every prime ideal of R is finitely generated so, by a theorem of Cohen, R is Noetherian. Since every nonzero prime ideal in R is maximal, it has Krull dimension 1. Moreover, since every maximal ideal of R is generated by a single element, the ring R is regular. It follows that R is a Dedekind ring, so that X_{FF} is a Dedekind scheme. □