Math 155, Problem Set 9 (due November 7)

November 6, 2011

(1) For every sequence of positive integers $m_1, \ldots, m_t$, let $R(m_1, \ldots, m_t)$ denote the corresponding Ramsey number: that is, the smallest integer n such that if G is a complete graph on n vertices, then every edge coloring of G using the set of colors $\{c_1, \ldots, c_t\}$ has monochromatic subgraph of size m_i and color c_i, for some $1 \leq i \leq t$. Prove that $R(m_1, \ldots, m_t) \leq R(m_1, \ldots, m_{t-2}, R(m_{t-1}, m_t))$. Use this to show that the 2-color version of Ramsey's theorem implies the many-color version.

In class, we proved two versions of van der Waerden's theorem:

Theorem 1 (Infinite Version)**.** *Let T be a finite set and let $f : \mathbf{Z} \to T$ be a T-coloring of the integers. Then there exist arbitrarily long monochromatic arithmetic progressions $S \subseteq \mathbf{Z}$.*

Theorem 2 (Finite Version)**.** *Let T be a finite set and let $k \geq 1$ be an integer. Then there exists a positive integer C with the following property: for every coloring $f : \{0, \ldots, C\} \to T$, there exists an monochromatic arithmetic progression $S \subseteq \{0, \ldots, C\}$ of size k.*

(2) Show directly that the infinite version of van der Waerden's theorem implies the finite version.

(3) Let G be a finite group and let $\mathrm{Burn}(G)$ be its Burnside ring. Let A be the collection of all subgroups of G, and regard A as a partially ordered set with respect to inclusions. Let μ denote the Möbius function of A. For each subgroup $H \subseteq G$, set

$$e_H = \sum_{H' \subseteq G} |H'| \mu(H', H)[G/H'] \in \mathrm{Burn}(G).$$

Prove that

$$e_{H_1} e_{H_2} = \begin{cases} |N(H_1)| e_{H_1} & \text{if } H_1 \text{ is conjugate to } H_2 \\ 0 & \text{otherwise.} \end{cases}$$

Here $N(H) = \{g \in G : g^{-1} H g = H\}$ denotes the normalizer of the group H. (Hint: Problem 3 on Problem Sets 6 and 8 may be helpful.)