Math 155, Problem Set 4 (due October 3)

September 25, 2011

(1) Using Polya's theorem, determine the number of isomorphism classes of graphs with four vertices and with five vertices. In the case of four vertices, verify your answer by giving a list of graphs, one from each isomorphism class.

(2) Let G be a finite group and let X be a finite G-set. For each subgroup $H \subseteq G$, let $X^H = \{x \in X : (\forall h \in H) hx = x\}$ denote the subset of X consisting of elements which are fixed by H. Let Y be another finite G-set. Prove that the following conditions are equivalent:

(i) There is an isomorphism of G-sets between X and Y.

(ii) The sets X^H and Y^H have the same number of elements, for each subgroup $H \subseteq G$.

(3) Let G be a finite group. Let $\mathrm{Burn}[G]$ denote the abelian group generated by symbols $[X]$, where X is a finite G-set, modulo the following relations:

- If X and Y are isomorphic G-sets, then $[X] = [Y]$.
- For every pair of finite G-sets X and Y, $[X \amalg Y] = [X] + [Y]$.

Prove that there is a unique commutative ring structure on $\mathrm{Burn}[G]$ satisfying the condition that $[X][Y] = [X \times Y]$ for every pair of finite G-sets X and Y. The commutative ring $\mathrm{Burn}[G]$ is called the *Burnside ring* of G.