Math 155, Problem Set 2 (due September 19)

September 11, 2011

(1) Prove the *trinomial theorem*, which asserts that for $n \geq 0$ we have an equality of polynomials

$$(x+y+z)^n = \sum_{p+q+r=n} \frac{n!}{p!q!r!} x^p y^q z^r.$$

(2) Define a power series in two variables by the formula

$$F(x,y) = \sum_{n,k \geq 0} \binom{n}{k} x^n y^k.$$

Translate the recurrence relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ into an identity satisfied by the power series $F(x,y)$, and use this to give a closed-form expression for $F(x,y)$.

(3) Let D_n denote the number of derangements of the set $\{1, 2, \ldots, n\}$. Using the formula established in class, prove the the integers D_n satisfy the recurrence relation

$$D_{n+1} = n(D_n + D_{n-1})$$

for $n \geq 1$. Then give direct combinatorial proof of this identity. (Hint: decompose the set of derangements σ of $\{1, \ldots, n+1\}$ into two subsets: those derangements which satisfy $\sigma^2(n+1) = n+1$, and those which do not).