Math 155 (Lecture 7)

September 16, 2011

In this lecture, we describe some applications of the composition formula

$$F_{S\circ T}(x) = F_S(F_T(x))$$

proved in the last lecture. We will be particularly interested in the case where $S = S_{\mathrm{Set}}$ is the species of finite sets, so that $S[I] = \{*\}$ for every finite set I and therefore $F_S(x) = e^x$. We will refer to $S \circ T$ as the *exponential* of the species T and denote it by $\exp(T)$, so that our formula reads

$$F_{\exp(T)}(x) = e^{F_T(x)}.$$

Example 1. Let S be the species of graphs: that is, $S[I]$ is the set of all graphs with vertex set I. Note that $S[I]$ is just the collection of all subsets of unordered pairs of elements of I, and therefore has cardinality $2^{\binom{|I|}{2}}$. We conclude that the exponential generating function of S is given by

$$F_S(x) = \sum_{n\geq 0} \frac{2^{\binom{n}{2}}}{n!} x^n = 1 + x + \frac{2}{2}x^2 + \frac{8}{6}x^3 + \frac{64}{24}x^4 + \cdots .$$

This function is unlikely to be familiar: the coefficients in the numerator grow faster than the coefficients in the denominator, so the power series does not converge for any nonzero value of x.

Let T be the species of *connected* graphs: that is, for each finite set I, $T[I]$ is the set of all connected graphs with vertex set I (by convention, we agree that the empty graph is not connected). Every graph with vertex set I can be uniquely decomposed into connected components. It follows that to give a graph with vertex set I, one must give an equivalence relation on I (the relation of "being in the same connected component") and, for each equivalence class, a connected graph with that vertex set. It follows that S is the exponential of the species T: we have an isomorphism of species $S = \exp(T)$ and therefore an equality of generating functions

$$F_S(x) = e^{F_T(x)}.$$

We can use this to solve for $F_T(x)$: we get

$$F_T(x) = \log F_S(x) = \log(1 + x + \frac{2}{2}x^2 + \frac{8}{6}x^3 + \cdots).$$

This does not obviously give us a closed form expression for $F_T(x)$ (that is, a closed form expression for the number of labelled connected graphs of a fixed size). But it does give us a reasonable approach to computing this number, which is much more efficient than trying to make a direct count.

Example 2. Let S be the species of permutations (so that $S[I]$ is the set of all permutation of I) and let S_0 be the species of *cyclic permutations*, so that $S_0[I]$ is the set of all permutations of I having only one orbit (by convention, the identity permutation of the empty set is not cyclic: it has zero orbits, rather than one). We already have a formula for the exponential generating function $F_S(x)$: it is given by

$$\sum_{n\geq 0} \frac{n!}{n!} x^n = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

Every permutation π of a set I determines an equivalence relation on I, given by "being in the same orbit". To specify the permutation π, we must give this equivalence relation together with a cyclic permutation of each equivalence class. We therefore have an isomorphism of species $S \simeq \exp(S_0)$, hence an equality of exponential generating functions

$$F_S(x) = e^{F_{S_0}(x)}.$$

It follows that

$$F_{S_0}(x) = \log F_S(x) = \log(\frac{1}{1-x}) = \sum_{n \geq 1} \frac{x^n}{n}.$$

From this, we deduce that if I is a finite set of size $n \geq 1$, then $S_0[I]$ has $\frac{n!}{n} = (n-1)!$ elements. In other words, there are exactly $(n-1)!$ cyclic permutations of the set $\{1, 2, \ldots, n\}$.

Example 2 is not the best illustration of the power of our method: it is fairly easy to show that there are $(n-1)!$ cyclic permutations of the set $\{1, 2, \ldots, n\}$ by a direct combinatorial argument (exercise). Here is a much trickier question:

Question 3. How many permutations of the set $\{1, \ldots, n\}$ have odd order? What is the probability that a randomly chosen permutation has odd order?

Let's denote the number of odd order permutations of the set $\{1, 2, \ldots, n\}$ by a_n.

Example 4. When $n = 0$ or $n = 1$, the only permutation of the set $\{1, \ldots, n\}$ is the identity, which has odd order. Thus $a_0 = a_1 = 1$.

The set $\{1, 2\}$ has two permutations, one of which has odd order and one of which does not. We therefore have $a_2 = 1$. The set $\{1, 2, 3\}$ has six permutations. Of these, the permutations of odd order include the identity and the two cyclic permutations, so that $a_2 = 3$.

The odd-order permutations of the set $\{1, 2, 3, 4\}$ include the identity permutation and those permutations which fix a single element and restrict to a 3-cycle on the remaining elements. Of the latter type, there are four choices for the fixed point and two choices for the cycle. We therefore have $a_4 = 1 + 2 \times 4 = 9$.

Odd order permutations of the set $\{1, 2, 3, 4, 5\}$ come in three types: cyclic permutations (of which there are $(5-1)! = 24$), permutations with two fixed points and a three-cycle (of which there are $2\binom{5}{2} = 20$), and the identity permutation. It follows that $a_5 = 24 + 20 + 1 = 45$.

It is a bit more instructive to list the probabilities $\frac{a_n}{n!}$: these are given by $1, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \ldots$ Here we might notice two patterns:

- (a) The sequence of probabilities $\frac{a_n}{n!}$ is non-increasing.
- (b) We have $\frac{a_{2n}}{(2n)!} = \frac{a_{2n+1}}{(2n+1)!}$.

We will use generating functions to show that both of these patterns persist. Let S be denote the species of odd-order permutations: that is, $S[I]$ is the set of all permutations of I having odd order, for each finite set I. Note that a permutation of I has odd order if and only if of its cycles has odd length. We may therefore write $S = \exp(S_0)$, where S_0 is the species of odd cycles: that is,

$$S_0[I] = \begin{cases} \{ \text{ cyclic permutations of I } \} & \text{if } |I| = 2k+1 \\ \emptyset & \text{if } |I| = 2k. \end{cases}$$

We have already seen that a set of size $2k + 1$ has exactly $2k!$ cyclic permutations. It follows that the exponential generating function of $F_{S_0}(x)$ is given by

$$\sum_{k \geq 0} \frac{2k!}{(2k+1)!} x^{2k+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots.$$

To obtain a closed formula for this, we take as our starting point the equality

$$\log \frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

Changing the sign of x, we also have

$$\log \frac{1}{1+x} = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots$$

Subtracting the second equation from the first, we get

$$\log \frac{1}{1-x} - \log \frac{1}{1+x} = 2F_{S_0}(x),$$

so that $F_{S_0}(x) = \frac{1}{2}(\log \frac{1}{1-x} + \log(1+x)) = \log \sqrt{\frac{1+x}{1-x}}$. It follows that

$$F_S(x) = e^{F_{S_0}(x)} = \sqrt{\frac{1+x}{1-x}} = \frac{\sqrt{1-x^2}}{1-x}.$$

To obtain a closed form expression for the numbers a_n, it is convenient to use the binomial formula

$$\begin{aligned} (1-x^2)^{1/2} &= \sum_{k \geq 0} \binom{\frac{1}{2}}{k} (-x^2)^k \\ &= \sum_{k \geq 0} \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!} (-1)^k x^{2k}. \end{aligned}$$

It follows that the coefficient $\frac{a_n}{n!}$ of x^n in $F_S(x) = \frac{1}{1-x}(1-x^2)^{\frac{1}{2}}$ is given by

$$\begin{aligned} \sum_{0 \leq k \leq \frac{n}{2}} \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-k+1)}{k!} (-1)^k &= 1 - \sum_{1 \leq k \leq \frac{n}{2}} \frac{(2k-3)(2k-5)\cdots(1)}{k! 2^k.} \\ &= 1 - \sum_{1 \leq k \leq \frac{n}{2}} \frac{(2k-3)!}{k!(k-2)! 2^{2k-2}} \end{aligned}$$

From this formula we can immediately read off properties (a) and (b): the range of the summation is the same for $n = 2m$ and $n = 2m+1$, and increasing n only increases the number of terms that need to be subtracted (each of which is positive).

Question 5. Can you find a direct combinatorial proof of assertions (a) and (b)?