Math 155 (Lecture 6)

September 13, 2011

In this lecture, we will continue our discussion of natural operations on species.

Definition 1. Let S and T be species. We define a new species ST, called the *product* of S and T, as follows:

(a) Let I be a finite set. We define $(ST)[I]$ to be the finite set given by the disjoint union

$$\coprod_{I=I_0\cup I_1} S[I_0]\times T[I_1].$$

Here the product is taken over all decompositions of I into disjoint subsets I_0 and I_1.

(b) If $\pi : I \to J$ is a bijection of finite sets, we take $(ST)[\pi] : (ST)[I] \to (ST)[J]$ to be the bijection given by the disjoint union of the maps

$$S[I_0]\times T[I_1] \to S[\pi(I_0)]\times T[\pi(I_1)]$$

determined by the bijections $I_0 \to \pi(I_0)$ and $I_1 \to \pi(I_1)$.

Example 2. Suppose we are given a finite set C of colors. For each finite set I, let $S^C[I]$ denote the set of all colorings of I by the set C: that is, the set C^I of all maps $I \to C$. Then S^C is a species, called the *species of coloring by* C.

Now suppose that C is given as a disjoint union of subsets C_0 and C_1. Then every coloring $f : I \to C$ of a set I determines a decomposition of I into disjoint subsets I_0 and I_1, given by

$$I_0 = f^{-1}C_0 \qquad I_1 = f^{-1}C_1.$$

Moreover, to recover the coloring of I, we need this decomposition together with a coloring of I_0 by C_0 and a coloring of I_1 by C_1. In other words, we have a bijection

$$S^C[I] = \coprod_{I=I_0\cup I_1} S^{C_0}[I_0]\times S^{C_1}[I_1].$$

These bijections give an *isomorphism* between the species S^C and the product species $S^{C_0}S^{C_1}$.

Proposition 3. *Let S and T be species with exponential generating functions $F_S(x)$ and $F_T(x)$. Then the product species ST has exponential generating function $F_{ST}(x) = F_S(x)F_T(x)$.*

Proof. We compute

$$\begin{aligned} F_S(x)F_T(x) &= (\sum_{p\geq 0} \frac{|S[\langle p\rangle]|}{p!}x^p)(\sum_{q\geq 0}\frac{|T[\langle q\rangle]|}{q!}x^q \\ &= \sum_{p,q\geq 0}\frac{|S[\langle p\rangle]|\,|T[\langle q\rangle]|}{p!q!}x^{p+q} \\ &= \sum_{n\geq 0}\sum_{p+q=n}\frac{n!}{p!q!}\frac{|S[\langle p\rangle]\times T[\langle q\rangle]|}{n!}x^n. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} F_{ST}(x) &= \sum_{n \geq 0} \frac{|(ST)[\langle n \rangle]|}{n!} x^n \\ &= \sum_{n \geq 0} \sum_{\langle n \rangle = I_0 \cup I_1} \frac{|S[I_0] \times T[I_1]|}{n!} x^n. \end{aligned}$$

Every decomposition $\langle n \rangle = I_0 \cup I_1$ into disjoint subsets determines a pair of natural numbers $p = |I_0|$, $q = |I_1|$ with $p + q = n$, and we have $|S[I_0] \times T[I_1]| = |S[\langle p \rangle] \times T[\langle q \rangle]|$. Each of these factors occurs precisely $\binom{n}{p} = \frac{n!}{p!q!} = \binom{n}{q}$ times in the second sum, so that

$$F_{ST}(x) = \sum_{n \geq 0} \sum_{p+q=n} \frac{n!}{p!q!} \frac{|S[\langle p \rangle] \times T[\langle q \rangle]|}{n!} x^n = F_S(x) F_T(x)$$

as desired. □

Example 4. Let C be a finite set with c elements, and let S^C be the species of C-colorings appearing in Example 2. Then $|S^C[I]| = |C^I| = c^{|I|}$ for every finite set I, so that

$$F_{S^C}(x) = \sum_{n \geq 0} \frac{c^n}{n!} x^n = e^{cx}.$$

If C is given as a disjoint union of subsets C_0 and C_1 having sizes c_0 and c_1, then we have an isomorphism of species $S^C \simeq S^{C_0} S^{C_1}$. Proposition 3 gives an identity of generating functions

$$F_{S^C}(x) = F_{S^{C_0}}(x) F_{S^{C_1}}(x),$$

which reduces to the familiar identity

$$e^{cx} = e^{c_0 x} e^{c_1 x}.$$

Example 5. Let S be the species of undecorated sets: $S[I]$ has a single element for every finite set I. The exponential generating function F_S is given by $\sum_{n \geq 0} \frac{1}{n!} x^n = e^x$. Let T be the species of derangements. The product ST assigns to each finite set I the collection all decompositions $I = I_0 \cup I_1$, together with a derangement of I_1. It follows that ST is isomorphic to the species of permutations (every permutation of the set I has a fixed point set $I_0 \subseteq I$, and determines a derangement of the complement $I_1 = I - I_0$). It follows that

$$\frac{1}{1-x} = F_{ST}(x) = F_S(x) F_T(x) = e^x F_T(x).$$

We therefore recover our formula

$$F_T(x) = \frac{e^{-x}}{1-x}$$

for the exponential generating function of derangements.

Definition 6. Let S and T be species, and assume that $T[\emptyset] = \emptyset$. We define a new species $S \circ T$ as follows. For each finite set I, let $(S \circ T)[I]$ denote the set of all triples $(\sim, x, \{y_J\}_{J \in I/\sim})$ where $\sim$ is an equivalence relation on I, $x \in S[I/\sim]$, and for each $J \in I/\sim$, y_J is an element of $T[J]$ (here we identify the elements of $I/\sim$ with equivalence classes in I).

Our goal in this section is to prove the following result:

Theorem 7. *Let S and T be species, and assume that $T[\emptyset] = \emptyset$. Then we have an equality of power series $F_{S \circ T}(x) = F_S(F_T(x))$. In other words, the construction $S \mapsto F_S$ is compatible with composition.*

Example 8. Let S be the species of nonempty sets: that is, we have

$$S[I] = \begin{cases} \{*\} & \text{if } I \neq \emptyset \\ \emptyset & \text{if } I = \emptyset. \end{cases}$$

For every finite set I, we see that $(S \circ S)[I]$ is the set of all equivalence relations $\sim$ on I such that each equivalence class is nonempty (this condition is automatic) and $I/\sim$ is nonempty (which is true if and only if I is nonempty). We therefore have

$$(S \circ S)[I] \simeq \begin{cases} \emptyset & \text{if } I = \emptyset \\ \text{partitions of } I & \text{if } I \neq \emptyset. \end{cases}$$

The exponential generating function for S is given by $F_S = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1$. It follows from Theorem 7 that the exponential generating function for $S \circ S$ is given by

$$F_{S \circ S}(x) = F_S(F_S(x)) = F_S(e^x - 1) = e^{e^x - 1} - 1.$$

From this, we recover our formula for the exponential generating function for the Bell numbers:

$$\sum_{n \geq 0} \frac{b_n}{n!} x^n = 1 + F_{S \circ S}(x) = 1 + e^{e^x - 1} - 1 = e^{e^x - 1}.$$

Warning 9. To make sense of Theorem 7, we need to define the compositon $F_S(F_T(x))$. This is given by

$$\sum_{n \geq 0} \frac{S[\langle n \rangle]}{n!} F_T(x)^n.$$

This sum is sensible because of our assumption that $T[\emptyset] = \emptyset$, which guarantees that the constant term of $F_T(x)$ vanishes (so that $F_T(x)^n$ is divisible by x^n).

Proof of Theorem 7. For each natural number $k \geq 0$, let X_k denote the species which assigns to each finite set I the subset $X_k[I] \subseteq (S \circ T)[I]$ consisting of those triples $(\sim, x, \{y_J\})$ where the quotient $I/\sim$ has exactly k elements. Then $(S \circ T)[I]$ is a disjoint union of the subsets $X_k[I]$ as k ranges over all natural numbers (note that the set $X_k[I]$ is empty if $k > |I|$), so that

$$F_{S \circ T}(x) = \sum_{k \geq 0} F_{X_k}(x).$$

Similarly, we have

$$F_S(F_T(x)) = \sum_{k \geq 0} \frac{S[\langle k \rangle]}{k!} F_T(x)^k.$$

We will complete the proof by showing that

$$F_{X_k}(x) = \frac{S[\langle k \rangle]}{k!} F_T(x)^k.$$

Let $\overline{X}_k$ denote the species which assigns to each finite set I the disjoint union

$$\coprod_{I = I_1 \cup \cdots \cup I_k} T[I_1] \times \cdots \times T[I_k] \times S[\langle k \rangle].$$

The sets $\overline{X}_k[I]$ and $X_k[I]$ are almost the same: the only difference is that an element of $X_k[I]$ specifies a partition of I into *unlabelled* pieces, while $\overline{X}_k[I]$ specifies a partition into *labelled* pieces. We therefore have

$$|\overline{X}_k[I]| = k! |X_k[I]|,$$

so that $F_{\overline{X}_k}(x) = k! F_{X_k}(x)$.

Using Proposition 3 repeatedly, we see that $F_T(x)^k$ is the exponential generating function for the species T^k. Unwinding the definitions, we see that T^k assigns to a finite set I the disjoint union

$$\coprod_{I = I_1 \cup \cdots \cup I_k} T[I_1] \times \cdots \times T[I_k],$$

where the coproduct is taken over all decompositions of I into disjoint labelled subsets $I_1, \ldots, I_k$. We therefore have

$$|\overline{X}_k[I]| = |T^k[I]| \, |S[\langle k \rangle]|$$

for each finite set I. Passing to generating functions, we get

$$F_T(x)^k |S[\langle k \rangle]| = F_{\overline{X}_k}(x) = k! F_{X_k}(x),$$

so that $F_{X_k}(x) = \frac{S[\langle k \rangle]}{k!} F_T(x)^k$ as desired. □