Math 155 (Lecture 5)

September 13, 2011

Our goal in this lecture is to introduce Joyal's theory of *species*, which provides a convenient language for formalizing many ideas in enumerative combinatorics.

Definition 1. A *species* is a rule S which does the following:

(a) To every finite set I, S assigns another finite set $S[I]$.

(b) To every bijection of finite sets $\pi : I \to J$, S assigns a bijection $S[\pi] : S[I] \to S[J]$.

Moreover, we assume that these bijections satisfy the following:

(c) If we are given finite sets I, J, and K, and bijections

$$\pi : I \to J \qquad \pi' : J \to K,$$

then $S[\pi' \circ \pi] = S[\pi'] \circ S[\pi]$.

Example 2 (Partitions)**.** To each finite set I, let $S[I]$ denote the set of all partitions of I into nonempty subsets. Then S is a species, called the *species of partitions*. Strictly speaking, to make S into a species, we need to specify the bijection $S[\pi] : S[I] \to S[J]$ associated to every bijection $\pi : I \to J$. Here we just do the natural thing: if we are given a partition $I = \bigcup I_\alpha$ of a set I, then $S[\pi]$ carries this to the induced partition $J = \bigcup \pi(I_\alpha)$ of the set J.

Example 2 illustrates an abuse of language which we will often engage in: to specify a species S, we will generally just describe the sets $S[I]$ where I is a finite set. It will generally be obvious how to define S on bijections between finite sets.

Example 3 (Subsets)**.** For each finite set I, let $S[I]$ denote the collection of subsets of I. Then S is a species, called the *species of subsets*.

Example 4 (Subsets of Size k)**.** For each finite set I, let $S[I]$ denote the collection of all k-element subsets of I. Then S is a species, called the *species of k-element subsets*.

Example 5 (Graphs)**.** For each finite set I, let $S[I]$ denote the set of all graphs having vertex set I. In other words, $S[I]$ is the collection of all symmetric, antireflexive relations on I (equivalently, $S[I]$ is the collection of all subsets of the collection of unordered pairs of elements of I). Then S is a species, called the *species of graphs*.

Example 6 (Connected Graphs)**.** For each finite set I, let $S[I]$ denote the set of all connected graphs having vertex set I. Then S is a species, called the *species of connected graphs*.

Example 7 (Linear Orderings)**.** For each finite set I, let $S[I]$ denote the set of all linear orderings of I: that is, the set of all ways to put the elements of I into an ordered list $\{i_1 < i_2 < \cdots < i_n\}$. If $\pi : I \to J$ is a bijection of finite sets, then π determines a bijection $S[\pi] : S[I] \to S[J]$, which carries an ordering $\{i_1 < i_2 < \cdots < i_n\}$ to the induced ordering $\{\pi(i_1) < \cdots < \pi(i_n)\}$ of the set J. Then S is a species, called the *species of linear orderings*.

Example 8 (Permutations)**.** For each finite set I, let $S[I]$ denote the set of all permutations of the set I. Then S is a species, called the *species of permutations*. Since this example can tend to be a little confusing, let us be explicit about the definition of S on bijections. If $\pi : I \to J$ is a bijection of finite sets, then every permutation $\sigma : I \to I$ of I determines a permutation of J, given by the formula $j \mapsto \pi(\sigma(\pi^{-1}(j)))$. We therefore have

$$S[\pi](\sigma) = \pi \circ \sigma \circ \pi^{-1}.$$

Example 9 (Derangements)**.** For each finite set I, let $S[I]$ denote the set of all derangements of the set I. Then S is a species, called the *species of derangements* (the definition of S on bijections is defined as in the previous example).

Remark 10. In condition (c) of Definition 1, we can take $I = J = K$ and $\pi = \pi' = \mathrm{id}$. We then learn that $S[\mathrm{id}] = S[\mathrm{id}] \circ S[\mathrm{id}]$: that is, S carries the identity map $\mathrm{id} : I \to I$ to the identity map $S[I] \to S[I]$.

Remark 11. Let S be a species, and let $\pi : I \to J$ be a bijection of finite sets. Then π has an inverse π^{-1} which satisfies $\pi \circ \pi^{-1} = \mathrm{id}_J$. It follows that

$$\mathrm{id}_{S[J]} = S[\mathrm{id}_J] = S[\pi \circ \pi^{-1}] = S[\pi] \circ S[\pi^{-1}].$$

From this we deduce that $S[\pi^{-1}] : S[J] \to S[I]$ is the inverse of the bijection $S[\pi] : S[I] \to S[J]$.

Remark 12. Definition 1 is naturally formulated using the language of category theory. There is a category $\mathcal{C}$ whose objects are finite sets and whose morphisms are given by bijections. A species is just a functor from the category $\mathcal{C}$ to itself.

Notation 13. For every natural number n, let $\langle n \rangle$ denote the finite set $\{1, 2, \ldots, n\}$ with n elements.

To describe a species S, we need to specify the sets $S[I]$ for every finite set I. Since I is finite, we can always choose a bijection $I \simeq \langle n \rangle$, where n is the number of elements of I. Since S is a species, this determines a bijection

$$S[I] \to S[\langle n \rangle].$$

Consequently, the behavior of S on arbitrary finite sets is determined by the sets $S[\langle n \rangle]$, where $n \geq 0$. However, these sets are not arbitrary. Every permutation π of the set $\{1, 2, \ldots, n\}$ determines a permutation $S[\pi]$ of the set $S[\langle n \rangle]$. Moreover, conditions (c) implies that for every pair of permutations π and π', we have $S[\pi' \circ \pi] = S[\pi'] \circ S[\pi]$. In other words, S determines an *action* of the group Σ_n of permutations of the set $\{1, \ldots, n\}$ on the set $S[\langle n \rangle]$. This leads to another formulation of Definition 1:

Definition 14. A *species* is a collection of sets $\{S_n\}_{n \geq 0}$, each of which is equipped with an action of the symmetric group Σ_n of permutations of the set $\{1, \ldots, n\}$.

Definitions 1 and 14 are essentially equivalent to one another. For example, given a species S in the sense of Definition 1, we can recover a species in the sense of Definition 14 by setting $S_n = S[\langle n \rangle]$. Definition 14 is a bit more concrete than Definition 1. However, Definition 1 will be a bit more convenient for us, and it is the definition that we will officially adopt.

Definition 15. Let S be a species. The *exponential generating function* of S is the power series

$$F_S(x) = \sum_{n \geq 0} \frac{|S[\langle n \rangle]|}{n!} x^n.$$

Example 16. In the last lecture, we saw that the species S of partitions has generating function $F_S(x) = e^{e^x - 1}$.

Example 17. Let S be the species of subsets. The set $\{1, \ldots, n\}$ has 2^n subsets, so that

$$F_S(x) = \sum_{n \geq 0} \frac{2^n}{n!} x^n = e^{2x}.$$

Example 18. Let S be the species of linear orderings. Note that there are exactly $n!$ linear orderings of a set of size n. It follows that

$$F_S(x) = \sum_{n \geq 0} \frac{n!}{n!} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Example 19. Let S be the species of permutations. Every set of size n has exactly $n!$ partitions, so that

$$F_S(x) = \sum_{n \geq 0} \frac{n!}{n!} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Warning 20. The species of linear orderings and the species of permutations have the same exponential generating function. However, when viewed as species, they are very different. Either species determines an action of the permutation group Σ_n on a set $S[\langle n \rangle]$ with $n!$ elements. In the case of linear orderings, this action is *free*: no linear ordering of the set $\{1, \ldots, n\}$ is preserved by any nontrivial permutation. However, the action of Σ_n on the set of permutations of $\{1, \ldots, n\}$ is *not free* (at least for $n \geq 2$): for example, it has the identity permutation as a fixed point.

One of the reasons that generating functions are so useful is that natural operations on power series (such as addition and multiplication) often have combinatorial interpretations in terms of species.

Example 21. Let S and T be species. We can define a new species $S + T$ as follows:

- (a) For every finite set I, we take $(S + T)[I]$ to be the disjoint union $S[I] \amalg S'[I]$ of the sets $S[I]$ and $T[I]$.
- (b) For every bijection $\pi : I \to J$, we let $(S + T)[\pi]$ denote the bijection from $S[I] \amalg T[I]$ to $S[J] \amalg T[J]$ determined by $S[\pi]$ and $T[\pi]$.

Since the size of a disjoint union of two sets is just the sum of their sizes, we have the following simple rule for exponential generating functions:

$$F_{S+T}(x) = F_S(x) F_T(x).$$

Example 22. Let S be a species. We define a new species S', called the *derivative of* S, as follows:

- (a) For every finite set I, we take $S'[I]$ to be the set $S[I \amalg \{*\}]$, obtained by applying the species S to the set $I \amalg \{*\}$ obtained by adding a single new element to I.
- (b) For every bijection $\pi : I \to J$, we take $S'[\pi]$ to be the bijection from $S'[I] = S[I \amalg \{*\}]$ to $S'[J] = S[J \amalg \{*\}]$ given by $S[\overline{\pi}]$, where $\overline{\pi} : I \amalg \{*\} \to J \amalg \{*\}$ is given by

$$\overline{\pi}(i) = \begin{cases} \pi(i) & \text{if } i \in I \\ * & \text{if } i = *. \end{cases}$$

For each $n \geq 0$, we have a bijection $S'[\langle n \rangle] \simeq S[\langle n+1 \rangle]$. It follows that the exponential generating function of S' is given by

$$\begin{aligned} F_{S'}(x) &= \sum_{n \geq 0} \frac{|S[\langle n+1 \rangle]|}{n!} x^n \\ &= \sum_{n \geq 0} |S[\langle n+1 \rangle]| (\frac{x^{n+1}}{(n+1)!})' \\ &= \sum_{m \geq 1} |S[\langle m \rangle]| (\frac{x^m}{m!})' \\ &= \sum_{m \geq 0} |S[\langle m \rangle]| (\frac{x^m}{m!})' \\ &= F'_S(x), \end{aligned}$$

where the superscript $'$ denotes differentiation of power series (with respect to the parameter x).