

Math 155 (Lecture 4)

September 8, 2011

Let us continue our analysis of the problem posed at the end of the previous lecture. For each $n \geq 0$, we let D_n denote the number of derangements of the set $\{1, \dots, n\}$. In the last lecture, we determined the exponential generating function

$$F(x) = \sum_{n \geq 0} \frac{D_n}{n!} x^n.$$

It is given by $F(x) = \frac{e^{-x}}{1-x}$.

Writing this out in more detail, we get

$$F(x) = \left(\sum_{p \geq 0} x^p \right) \left(\sum_{q \geq 0} \frac{(-1)^q x^q}{q!} \right).$$

Taking the coefficient of x^n on both sides, we get

$$\frac{D_n}{n!} = \sum_{p+q=n} \frac{(-1)^q}{q!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{6!} + \dots + \frac{(-1)^n}{n!}.$$

Clearing denominators, we get

$$D_n = \sum_{0 \leq q \leq n} \frac{(-1)^q n!}{q!} = n! - n! + \frac{n!}{2!} - \frac{n!}{3!} + \dots + (-1)^n.$$

Example 1. Our formula checks out for small values of n : we have

$$D_0 = 0! = 1 \quad D_1 = 1! - 1! = 0$$

$$D_2 = 2! - 2! + \frac{2!}{2!} = 1 \quad D_3 = 3! - 3! + \frac{3!}{2!} - \frac{3!}{3!} = 6 - 6 + 3 - 1 = 2.$$

This gives us the desired formula for the number of derangements, but gives an even nicer answer to our second formulation of the question: what is the probability that a given permutation is a derangement? The answer is given by

$$\frac{D_n}{n!} = \sum_{0 \leq q \leq n} \frac{(-1)^q}{q!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}$$

Note that this expression is the beginning of the power series expansion for the exponential function e^x , evaluated at $x = -1$. We therefore have

$$\frac{D_n}{n!} + \sum_{q > n} \frac{(-1)^q}{q!} = \sum_{q \geq 0} \frac{(-1)^q}{q!} = \frac{1}{e}.$$

The second summand here is very small: it is bounded in absolute value by $\frac{1}{(n+1)!}$. We conclude that $\frac{D_n}{n!}$ is very close to the real number $\frac{1}{e}$. In fact, this even gives an approximate formula for D_n itself: we see that

$$|D_n - \frac{n!}{e}| < \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

For $n \geq 1$, this means that D_n is the nearest integer to the expression $\frac{n!}{e}$.

Let us call attention to the following point: in order to get our nice formula for $F(x)$, it was essential to add the extra factor of $\frac{1}{n!}$ on the coefficient of x^n . The power series

$$G(x) = \sum_{n \geq 0} D_n x^n$$

is not going to be any recognizable function: for example, it does not converge for any nonzero value of x (because the sequence of coefficients $D_n \approx \frac{n!}{e}$ grows faster than any exponential function of n). This raises a question: how should one know to think about the exponential generating function F , rather than the ordinary generating function G ? (Put another way: why do the probabilities $\frac{D_n}{n!}$ behave more nicely than the integers D_n themselves?) A general rule of thumb is that exponential generating functions are useful for counting the number of solutions to some combinatorial problem involving a *labelled* set of size n .

Example 2. Let $n = 2m$ be an even integer. Let us say that a *labelled matching* of the set $\{1, \dots, n\}$ is a partition of this set into a sequence of m subsets S_1, \dots, S_m , each of size 2 (here the order of the sets S_i matters). Let M_n denote the number of labelled matchings of the set $\{1, \dots, n\}$.

By convention, we will agree that $M_n = 0$ if n is odd. If $n = 2m$ is even, then there are $\binom{n}{2} = \frac{n(n-1)}{2}$ choices for the subset S_1 . Once S_1 has been chosen, there are $\binom{n-2}{2} = \frac{(n-2)(n-3)}{2}$ choices for the subset S_2 , and so forth. The total number of labelled machings is then given by

$$M_n = \frac{n(n-1)}{2} \frac{(n-2)(n-3)}{2} \dots \frac{(2)(1)}{2} = \frac{n!}{2^{\frac{n}{2}}}.$$

If we try to organize these numbers into an ordinary generating function, we obtain the power series

$$\sum_{n \geq 0} \begin{cases} \frac{n!}{2^{\frac{n}{2}}} x^n & n \text{ even} \\ 0 & n \text{ odd,} \end{cases}$$

which is not convergent for any positive value of x (and therefore is not likely to be a power series which is familiar from calculus). However, the exponential generating function

$$\sum_{n \geq 0} \frac{M_n}{n!} x^n$$

is easy to describe: it is given by

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{8} + \dots = \frac{1}{1 - \frac{x^2}{2}} = \frac{2}{2 - x^2}.$$

One of the reasons that generating functions are so useful is that natural operations on power series (such as addition and multiplication) often have combinatorial interpretations. Let us consider the case of multiplication. Suppose we are given two sequences of integers $\{A_p\}_{p \geq 0}$ and $\{B_q\}_{q \geq 0}$, and consider the exponential generating functions

$$F(x) = \sum_{p \geq 0} \frac{A_p}{p!} x^p \quad G(x) = \sum_{q \geq 0} \frac{B_q}{q!} x^q,$$

and let $F(x)G(x) = \sum_{n \geq 0} \frac{C_n}{n!} x^n$. We have

$$\begin{aligned} F(x)G(x) &= \left(\sum_{p \geq 0} \frac{A_p}{p!} x^p \right) \left(\sum_{q \geq 0} \frac{B_q}{q!} x^q \right) \\ &= \sum_{p, q \geq 0} \frac{A_p B_q}{p! q!} x^{p+q} \\ &= \sum_{n \geq 0} \left(\sum_{p+q=n} \frac{A_p B_q}{p! q!} \right) x^n \end{aligned}$$

so that

$$C_n = n! \sum_{p+q=n} \frac{A_p B_q}{p! q!} = \sum_{0 \leq p \leq n} \binom{n}{p} A_p B_{n-p}.$$

Now suppose that the integers A_p are given by the solution to some counting problem: A_p is the number of ways to endow a set of size p with some kind of decoration. Similarly, suppose that B_q is the number of ways to endow a set of size q with some other kind of decoration. Then C_n has the following interpretation: it is the number of ways to partition a set of size n into two pieces, to endow the first piece with the first type of decoration, and to endow the second piece with the second type of decoration.

Example 3. Suppose that C_n is the number of ways to color the set $\{1, \dots, n\}$ using $c + c'$ colors. We have already encountered this number in the first lecture: it is $(c + c')^n$. Every coloring determines a partition of the set $\{1, \dots, n\}$ into two parts: a part where we have used the first c colors, and a part where we have used the remaining c' colors. If we let A_p denote the number of ways to color a set of size p with c colors and B_q the number of ways to color a set of size q with c' colors, the same analysis gives

$$A_p = c^p \quad B_q = c'^q.$$

In this case, we recover the binomial formula

$$(c + c')^n = C_n = \sum_{0 \leq p \leq n} \binom{n}{p} A_p B_{n-p} = \sum_{0 \leq p \leq n} \binom{n}{p} c^p c'^{n-p}.$$

At the level of generating functions, we have

$$\begin{aligned} \sum \frac{A_p}{p!} x^p &= e^{cx} \\ \sum \frac{B_q}{q!} x^q &= e^{c'x} \\ \sum \frac{C_n}{n!} x^n &= e^{(c+c')x}, \end{aligned}$$

and we recover the usual formula

$$e^{(c+c')x} = e^{cx} e^{c'x}.$$

Let's apply this to a more interesting problem.

Question 4. How many ways can the set $\{1, \dots, n\}$ be partitioned into nonempty subsets?

The answer to Question 4 is called the *Bell number* b_n . Note that Question 4 is similar to a question raised in the second lecture, in the definition of the Stirling numbers $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$. The only difference is that we have not specified the number k ahead of time. We therefore have

$$b_n = \sum_k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}.$$

Example 5. The first few Bell numbers are easy to compute by hand. We have $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, $b_3 = 5$, $b_4 = 15$.

Let's work out the exponential generating function $F(x) = \sum_{n \geq 0} \frac{b_n}{n!} x^n$ for the Bell numbers, using the multiplication principle introduced above. As a first step, let's isolate a recurrence relation satisfied by the Bell numbers. Suppose the integers b_0, b_1, \dots, b_n are known. How can we compute b_{n+1} without explicitly listing all the partitions of the set $\{1, \dots, n+1\}$? Note that every partition of $\{1, \dots, n+1\}$ determines a decomposition of the set $\{1, \dots, n\}$ into two parts those elements which are grouped with $n+1$, and those elements which are not. The first of these can be an arbitrary subset $S \subseteq \{1, \dots, n\}$, having some size k . To recover our original partition, we need to know the set S together with the resulting partition of $\{1, \dots, n\} - S$, and there are b_{n-k} choices for the latter. This analysis gives

$$b_{n+1} = \sum_{0 \leq k \leq n} \binom{n}{k} b_{n-k}.$$

Multiplying by $\frac{x^n}{n!}$ and summing over n , we get

$$\sum_{n \geq 0} \frac{b_{n+1}}{n!} x^n = \sum_{p \geq 0} \frac{1}{p!} x^p \sum_{q \geq 0} \frac{b_q}{q!} x^q = e^x F(x).$$

Note that the left hand side is just given by the *derivative* of the function $F(x)$. We deduce that F satisfies the differential equation

$$F'(x) = e^x F(x).$$

This differential equation is just another way of writing our recurrence relation above. Consequently, we see that the solution to this equation is uniquely determined provided that the constant term $F(0) = b_0 = 1$ has been specified. This solution is given by

$$F(x) = e^{e^x - 1} = \frac{1}{e} e^{e^x}$$

We have a convergent power series expansion

$$\begin{aligned} e^{e^x} &= \sum_{m \geq 0} \frac{1}{m!} (e^x)^m \\ &= \sum_{m \geq 0} \frac{1}{m!} e^{mx} \\ &= \sum_{m \geq 0, n \geq 0} \frac{1}{m!} \frac{1}{n!} (mx)^n. \end{aligned}$$

It follows that the coefficient of x^n in e^{e^x} is given by

$$\frac{1}{n!} \sum_{m \geq 0} \frac{m^n}{m!}.$$

We have proven:

Theorem 6 (Dobinski's formula). *The Bell numbers are given by*

$$b_n = \frac{1}{e} \sum_{m \geq 0} \frac{m^n}{n!}.$$