Math 155 (Lecture 33)

November 27, 2011

Definition 1. Let G be a graph with vertex set V. We say that G is *bipartite* if there exists a decomposition $V = V_0 \cup V_1$ into disjoint subsets, such that every edge consists of a vertex from V_0 together with a vertex from V_1. (That is, neither V_0 nor V_1 contains a pair of adjacent vertices.)

Remark 2. A disjoint union of bipartite graphs is bipartite.

Proposition 3. *Let G be a graph. The following conditions are equivalent:*

(1) *The graph G is bipartite.*

(2) *The graph G contains no cycles of odd length.*

Proof. Suppose first that G is bipartite, and let $v_0, v_1, \ldots, v_n = v_0$ be a cycle of G. Since G is bipartite, its vertex set V can be partitioned into subsets V_0 and V_1 as in Definition 1. We may assume without loss of generality that $v_0 \in V_0$. Then v_1 is adjacent to v_0, so that $v_1 \in V_1$. The same argument shows that $v_2 \in V_0$, $v_3 \in V_1$, and so forth. Since $v_n = v_0 \in V_0$, we conclude that n is even.

Now suppose that condition (2) is satisfied. We wish to show that G is bipartite. Since the collection of bipartite graphs is closed under disjoint unions (Remark 2), we may suppose also that G is connected. Fix a vertex $v \in V$. For each $w \in V$, let $d(v, w)$ denote the distance from v to w: that is, the length of the shortest path from v to w. Let $V_0 = \{w \in V : d(v, w) \text{ is even }\}$ and $V_1 = \{w \in V : d(v, w) \text{ is odd }\}$. We claim that the decomposition $V = V_0 \cup V_1$ satisfies the requirements of Definition 1. Suppose otherwise: then there exists a pair of adjacent vertices $w, w' \in V$ such that either $w, w' \in V_0$ or $w, w' \in V_1$. Let us assume that $w, w' \in V_0$ (the other case can be handled in a similar way). Then there exists a path $v = v_0, v_1, \ldots, v_m = w$ of even length. Similarly, there exists a path $v = v'_0, v'_1, \ldots, v'_n = w'$ of even length. Then the cycle $v = v_0, v_1, \ldots, v_m = w, w' = v'_n, v'_{n-1}, \ldots, v'_0 = v$ has length $m + n + 1$ which is an odd number, contradicting assumption (2). □

Now suppose that G is a bipartite graph with vertex set V, and that we are given a decomposition $V = V_0 \cup V_1$ satisfying the requirements of Definition 1. A *matching* of V_0 to V_1 is an injective map $f : V_0 \to V_1$ such that $f(v)$ is adjacent to v, for each $v \in V_0$.

Question 4 (Marriage Problem)**.** Given a bipartite graph G as above, when does there exists a matching $f : V_0 \to V_1$?

Remark 5. The terminology of Question 4 is motivated as follows: we imagine that V_0 is the set of men in some village and V_1 the set of women in some village, and that a pair of vertices $v \in V_0$, $w \in V_1$ are adjacent if they are willing to marry. Then Question 4 asks if some matchmaker could arrange a marriage for every man in the village, with no two men marrying the same woman.

Remark 6. As formulated in Question 4, the marriage problem is not symmetric. However, if V_0 and V_1 have the same size, then any injection from V_0 to V_1 is a bijection, whose inverse is an injection from V_1 to V_0. Thus, in this special case, the problem is symmetric.

There are some situations which one can obviously not solve the marriage problem of Question 4:

Example 7. If $|V_0| > |V_1|$, then there cannot exist any map $f : V_0 \to V_1$. It follows that there cannot be a matching between V_0 and V_1.

Example 8. If there is some vertex $v \in V_0$ which is not adjacent to any vertex in V_1, then there cannot be a matching from V_0 to V_1.

We can simultaneously rule out the bad situations described in Examples 7 and 8 with the following assumption:

($*$) For every subset $S \subseteq V_0$, let $S^+ \subseteq V_1$ be the set $\{v \in V_1 : v \text{ is adjacent to some } w \in S\}$. Then $|S^+| \geq |S|$.

When S has a single element, this says that every vertex of V_0 is adjacent to some vertex of V_1. When $S = V_0$, it guarantees that $|V_1| \geq |V_0|$.

Theorem 9 (Hall's Marriage Theorem)**.** *Let G be a bipartite graph with vertex set $V = V_0 \cup V_1$ as above. Then there is a matching $f : V_0 \to V_1$ if and only if condition ($*$) is satisfied.*

Proof. We first prove the "only if" direction. Suppose there is a matching $f : V_0 \to V_1$, and let $S \subseteq V_0$. Then $f(S) \subseteq S^+$, so that $|S^+| \geq |f(S)| = |S|$.

The hard part is to prove the "if" direction. We will prove, using induction on the integer $|V_0|$, that condition ($*$) implies the existence of a matching $V_0 \to V_1$. We consider two cases:

(a) Suppose that there exists a nonempty proper subset $S \subset V_0$ such that $|S^+| = |S|$. Applying the inductive hypothesis, we can find a matching $f : S \to S^+$. Let G' be the graph obtained from G by removing S and $f(S)$, so that the set of vertices of G' can be decomposed into subsets $W_0 = V_0 - S$ and $W_1 = V_1 - f(S)$. For each subset $T \subseteq W_0$, let $T^+ \subseteq W_1$ be defined as in ($*$). Then $(S \cup T)^+ \subseteq T^+ \cup S^+ = T^+ \cup f(S)$, so that $|T^+| = |(S \cup T)^+| - |f(S)| \geq |S \cup T| - |S| = |T|$. It follows that the graph G' satisfies condition ($*$), so that the inductive hypothesis guarantees the existence of a matching $g : W_0 \to W_1$. Together, the maps f and g determine a matching $V_0 \to V_1$.

(b) Suppose that for *every* nonempty proper subset $S \subseteq V_0$, we have $|S^+| > |S|$. If V_0 is empty, there is nothing to prove. Otherwise, we can choose a vertex $v \in V_0$. Since $\{v\}^+$ is nonempty, we can choose a vertex $w \in V_1$ adjacent to v. Let G' be the graph obtained from G by removing the vertices v and w, so that the vertices of G' can be decomposed into subsets $W_0 = V_0 - \{v\}$ and $W_1 = V_1 - \{w\}$. For each nonempty subset $S \subseteq W_1$, the set $S^{++} = \{u \in W_1 : u \text{ is adjacent to some } t \in T\}$ coincides with $S^+ - \{w\}$, where S^+ is computed in the graph G. Since S is a proper subset of V_0, we have $|S^{++}| \geq |S^+| - 1 > |S| - 1$, so that $|S^{++}| \geq |S|$. Thus the graph G' satisfies ($*$), so the inductive hypothesis gives us a matching $g : W_0 \to W_1$. This extends to a matching $f : V_0 \to V_1$ by setting $f(v) = w$.

□

Here is a reformulation of the marriage theorem which does not mention graphs:

Theorem 10. *Let X be a finite set, and suppose we are given subsets $Y_1, Y_2, \ldots, Y_m \subseteq X$. Assume that:*

($*'$) *For every subset $S \subseteq \{1, \ldots, m\}$, the set $\bigcup_{i \in S} Y_i$ has cardinality at least $|S|$.*

Then there exists a sequence of elements $y_1 \in Y_1$, $y_2 \in Y_2$, $\ldots$, with $y_i \neq y_j$ for $i \neq j$.

Proof. Form a bipartite graph G with vertex set $X \cup \{1, \ldots, m\}$, where an element $x \in X$ is adjacent to $i \in \{1, \ldots, m\}$ if $x \in Y_i$. Condition ($*'$) implies that G satisfies hypothesis ($*$) of Hall's marriage theorem, so that there exists a matching $f : \{1, \ldots, m\} \to X$. Now set $y_1 = f(1)$, $y_2 = f(2)$, and so forth. □

In the symmetric case $|V_0| = |V_1|$, Question 4 can be regarded as a special case of a more general question.

Definition 11. Let G be a graph. A *matching* of G is a set M of edges of G, no two of which share a vertex. We say that a matching is *perfect* if every vertex of G belongs to some edge of M.

Question 12. Given a graph G, when does it have a perfect matching?

An answer is provided by the following result:

Theorem 13 (Tutte)**.** *Let G be a finite graph with vertex set V. Then G has a perfect matching if and only if the following condition is satisfied, for every subset $S \subseteq V$:*

($\star$) *Let G' be the graph obtained from G by removing the set S. Then the number connected components of G' of odd size is $\leq |S|$.*

Example 14. When $S = \emptyset$, condition ($\star$) asserts that G has no components with an odd number of vertices. In particular, this implies that the number of vertices of G is even.

To prove the necessity of condition ($\star$), let us suppose that G has a perfect matching M. Let S be a set of vertices of G and let G' be as in ($\star$). If G'' is a connected component of G' with an odd number of vertices, then G' does not admit a perfect matching. Consequently, there exists at least one edge belonging to M which connects a vertex of G'' with one of the vertices of S. The vertices of S which arise in this way are all distinct (since two edges of M cannot share of a vertex). Consequently, the number of odd components of G' must be $\leq |S|$.

We will prove the sufficiency of condition ($\star$) in the next lecture.