Math 155 (Lecture 27)

November 6, 2011

In this lecture, we will discuss two other Ramsey-type results: the Hales-Jewett Theorem and van der Waerden's theorem.

We first consider the Hales-Jewett theorem. Informally speaking, it asserts the following: in large enough dimensions, the game of Tic-Tac-Toe cannot end in a draw. Let us explain this more precisely. For each positive integer k, let $\langle k \rangle = \{1, \ldots, k\}$. For each integer n, consider the n-dimensional cubical array $\langle k \rangle^n$. The elements $x \in \langle k \rangle^n$ can be identified with sequences $x(1), x(2), \ldots, x(n) \in \langle k \rangle$.

We will say that a sequence of elements $x_1, \ldots, x_k \in \langle k \rangle^n$ form a *line* if the following conditions are satisfied:

(a) For each $1 \leq j \leq n$, either the sequence $x_1(j), x_2(j), \cdots, x_k(j) \in \langle k \rangle$ is constant, or we have $x_i(j) = i$ for each $1 \leq i \leq k$.

(b) The sequence $x_1, \ldots, x_k$ is not constant (in other words, for at least one value $1 \leq j \leq n$, the second case of (a) holds).

In this case, we will also say that the set $\{x_1, \ldots, x_k\} \subseteq \langle k \rangle^n$ is a line.

Theorem 1 (Hales-Jewett)**.** *Let $T = \{c_1, \ldots, c_t\}$ be a finite set of colors, and let $k \geq 1$ be a positive integer. Then there exists a positive integer n with the following property: for every coloring*

$$f : \langle k \rangle^n \to T$$

of an n-dimensional cubical array, there exists a monochromatic line: that is, a line $L \subseteq \langle k \rangle^n$ such that f is constant on L.

Remark 2. In the situation of Theorem 1, we will denote the smallest possible value of n by $HJ(k, t)$.

Example 3. If $t = 1$, then there is a unique coloring of $\langle k \rangle^n$ and every line in $\langle k \rangle^n$ is monochromatic. We therefore have $HJ(k, 1) = 1$.

Example 4. If $k = 2$, then every two-element subset of $\langle k \rangle^n$ is a line. It follows that an integer n satisfies the conclusions of Theorem 1 if and only if every map $\langle k \rangle^n \to T$ takes on some value more than once. This is true if and only if $2^n > t$. It follows that $HJ(2, t)$ is the smallest integer n such that $2^n > t$.

Example 5. Since the usual 2-dimensional version of Tic-Tac-Toe sometimes (usually?) ends in a draw, we have $HJ(3, 2) > 2$.

Let us now prove Theorem 1. The proof will proceed by induction on k. If $k \leq 2$, the result is easy (Example 4); we may therefore assume that $k \geq 3$. For a fixed value of k, we will prove the result by induction on t. If $t = 1$, there is nothing to prove (Example 3). We may therefore assume $t > 1$. To prove the existence of the integer $HJ(k, t)$, we may assume the following:

(a) There exists an integer $m = HJ(k, t-1)$ such that every coloring of $\langle k \rangle^m$ using $t - 1$ colors has a monochromatic line.

(b) For every integer q, there exists an integer $n_q = HJ(k-1, q)$ such that every coloring of $\langle k-1\rangle^{n_q}$ using q colors has a monochromatic line.

Let us see where (b) gets us. Take $q = t$, and take $C_0 = n_t$. Every coloring of $\langle k\rangle^{C_0}$ using t colors gives in particular a coloring of $\langle k-1\rangle^{C_0} \subseteq \langle k\rangle^{C_0}$ with t colors. Invoking (b), we see that this second coloring has a monochromatic line $\{x_1, x_2, \ldots, x_{k-1}\}$. Note that this line extends uniquely to a line $L_0 = \{x_1, x_2, \ldots, x_k\} \subseteq \langle k\rangle^{C_0}$. However, we are not guaranteed that this larger line is monochromatic.

Let $C_1 = HJ(k-1, t^{k^{C_0}})$, and suppose we are given a coloring $f : \langle k\rangle^{C_0+C_1} \to T$ with $|T| = t$. Let $T(1)$ denote the set of all colorings of $\langle k\rangle^{C_0}$ by T. We have a canonical bijection of sets

$$\langle k\rangle^{C_0+C_1} \simeq \langle k\rangle^{C_0} \times \langle k\rangle^{C_1}.$$

We may therefore identify f with a map $\langle k\rangle^{C_1} \to T(1)$: that is, with a coloring of $\langle k\rangle^{C_1}$ with $|T(1)| = t^{k^{C_0}}$ colors. In particular, this gives a coloring of $\langle k-1\rangle^{C_1}$ with $t^{k^{C_0}}$ colors. Invoking (b), we see that there exists a monochromatic line in $\langle k-1\rangle^{C_1}$ which extends uniquely to a line $L_1 = \{y_1, \ldots, y_k\} \subseteq \langle k\rangle^{C_1}$. By construction, there is a fixed element $c \in T(1)$ such that y_i is assigned the color g for each $i < k$. We can regard g as a coloring $\langle k\rangle^{C_0} \to T$, and our choice of C_0 guarantees us a line $L_0 = \{x_1, \ldots, x_k\} \subseteq \langle k\rangle^{C_0}$ such that $x_1, \ldots, x_{k-1}$ are assigned the same color c. Let us identify $L_0 \times L_1$ with a subset of $\langle k\rangle^{C_0+C_1}$. The elements of $L_0 \times L_1$ are given by ordered pairs (x_i, y_j). Then $f(x_i, y_j) = c$ whenever $i, j < k$.

Now let $C_2 = HJ(k-1, t^{k^{C_0+C_1}})$. Suppose we are given a coloring of $f : \langle k\rangle^{C_0+C_1+C_2} \to T$ with $|T| = t$. Let $T(2)$ denote the set of all colorings by $\langle k\rangle^{C_0+C_1}$ by T. Arguing as before, we can identify f with a coloring $\langle k\rangle^{C_2} \to T(2)$. Since $|T(2)| = t^{k^{C_0+C_1}}$ which determines a coloring $\langle k-1\rangle^{C_2} \to T(2)$. Invoking (b), we can choose a monochromatic line in $\langle k-1\rangle^{C_2}$, which extends uniquely to a line $L_2 = \{z_1, \ldots, z_k\} \subseteq \langle k\rangle^{C_2}$. By construction, the elements z_i for $i < k$ are assigned the same color $g \in T(2)$, which we can identify with a coloring $\langle k\rangle^{C_0+C_1} \to T$. Applying the argument of the preceding paragraph, we can find a pair of lines $L_0 = \{x_1, \ldots, x_k\} \subseteq \langle k\rangle^{C_0}$ and $L_1 = \{y_1, \ldots, y_k\} \subseteq \langle k\rangle^{C_1}$ and a color $c \in T$ such that $g(x_i, y_{i'}) = c$ whenever $i, i' < k$. It follows that $f(x_i, y_{i'}, z_{i''}) = c$ whenever $i, i', i'' < k$.

Proceeding in this way, we recursively define integers C_j by the formula $C_j = HJ(k-1, t^{k^{C_0+\cdots+C_{j-1}}})$. The argument above shows that for every coloring

$$f : \langle k\rangle^{C_0+\cdots+C_j} \to T,$$

we can choose a color $c \in T$ and lines $L_i = \{x(i)_1, \ldots, x(i)_k\} \subseteq \langle k\rangle^{C_i}$ such that

$$f(x(0)_{i_0}, x(1)_{i_1}, \ldots, x(j)_{i_j}) = c$$

provided that $i_0, i_1, \ldots, i_j < k$.

Now let $m = HJ(k, t-1)$ be as in (a). We claim that the integer $C_0 + C_1 + \cdots + C_m$ satisfies the requirements of Theorem 1: that is, we have $HJ(k, t) \leq C_0 + \cdots + C_m$. To prove this, suppose we are given a coloring

$$f : \langle k\rangle^{C_0+\cdots+C_m} \to T,$$

and choose a color $c \in T$ and lines $L_i \subseteq \langle k\rangle^{C_i}$ as above. Write $L_0 = \{x(0)_1, \ldots, x(0)_k\}$, and let g be the restriction of f to the subset

$$\{x(0)_k\} \times L_1 \times \cdots \times L_m \subseteq \langle k\rangle^{C_0+C_1+\cdots+C_m}$$

There are two cases to consider:

(i) Suppose that the coloring g never takes the value c. Then we can identify g with a coloring $\langle k\rangle^m \to T - \{c\}$. Since $m = HJ(k, t-1)$, we conclude that $\langle k\rangle^m$ has a monochromatic line (of color $\neq c$), which is also a monochromatic line for f.

(ii) Suppose that g takes the value c somewhere. That is, there exist integer $i_1, i_2, \ldots, i_m \leq k$ such that

$$f(x(0)_k, x(1)_{i_1}, \ldots, x(m)_{i_m}) = c.$$

For $1 \leq p \leq m$ and $1 \leq q \leq k$, let

$$y(p)_q = \begin{cases} x(p)_q & \text{if } x(p)_{i_q} \neq x(p)_k \\ x(p)_{i_q} & \text{otherwise.} \end{cases}$$

and set $y_q = (x(0)_q, y(1)_q, \ldots, y(m)_q)$. Then $\{y_1, y_2, \ldots, y_k\}$ is a monochromatic line in $\langle k \rangle^{C_0 + \cdots + C_m}$ having color c.

Remark 6. The proof of Theorem 1 presented above gives an upper bound for the integers $HJ(k,t)$. However, the numbers which come out of the proof sketched above quickly become astronomically large. Using more refined arguments, one can obtain more reasonable upper bounds.

Let us now describe an application of the Hales-Jewett theorem.

Theorem 7 (van der Waerden)**.** *Let T be a finite set with t elements, and let $k \geq 1$ be an integer. There there exists a positive integer C with the following property: for every coloring $\{0, \ldots, C\} \to T$, there exists an arithmetic progression $S \subseteq \{0, \ldots, C\}$ of size at least k.*

Proof. Let $n = HJ(k,t)$, and let $C = k^n - 1$. We have a canonical bijection

$$\phi : \langle k \rangle^n \to \{0, \ldots, C\},$$

given by $(x_1, \ldots, x_n) \mapsto \sum (x_i - 1) k^{i-1}$ (the inverse bijection just assigns to each integer $0 \leq p < k^n$ the digits in its base n expansion). Every coloring $\{0, \ldots, C\} \to T$ determines a coloring

$$\langle k \rangle^n \to T.$$

Since $n = HJ(k,t)$, for every such coloring there is a monochromatic line $L \subseteq \langle k \rangle^n$. It now suffices to observe that $\phi(L)$ is an arithmetic progression of length k. $\square$

Corollary 8 (Infinite Version of van der Waerden's Theorem)**.** *Let T be a finite set, and let $f : \mathbf{Z} \to T$ be a coloring of the integers. Then there exist arbitrarily large arithmetic progressions $S \subseteq \mathbf{Z}$ such that $f|S$ is constant (that is, monochromatic arithmetic progressions).*