

Math 155 (Lecture 26)

November 3, 2011

Our first goal in this lecture is to prove the following infinite version of Ramsey's theorem:

Theorem 1. *Let G be a complete graph with infinitely many vertices, and suppose we are given an edge coloring of G using a finite set $T = \{c_1, c_2, \dots, c_t\}$ of colors. Then G has an infinite monochromatic subgraph.*

Corollary 2. *Let G be an infinite graph. Then G either contains an infinite clique or an infinite anticlique.*

Proof of Theorem 1. Choose a vertex $v_0 \in G$. There are infinitely many vertices of G different from v_0 , and only finitely many colors to choose from. It follows that there exists an infinite subgraph $G_1 \subseteq G$ such that each vertex of G_1 is connected to v_0 by an edge of the same color, which we will denote by c_{i_0} .

Choose a vertex $v_1 \in G_1$. There are infinitely many vertices of G_1 different from v_1 , and only finitely many colors to choose from. It follows that we can choose an infinite subgraph $G_2 \subseteq G_1$ such that each vertex of G_2 is connected to v_1 by an edge of the same color, which we will denote by c_{i_1} .

Continuing in this way, we can an infinite sequence of subgraphs

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots,$$

colors $c_{i_0}, c_{i_1}, \dots \in T$, and vertices $v_j \in G_j$ with the following property: for each vertex $w \in G_{j+1}$, the edge connecting v_j to w has the color c_{i_j} . In particular, if $j < k$, then the edge connecting v_j to v_k has color c_{i_j} .

In the infinite list of colors

$$c_{i_0}, c_{i_1}, c_{i_2}, \dots,$$

some color must occur infinitely many times. We may therefore choose a sequence of integers j_0, j_1, \dots such that

$$c_{i_{j_0}} = c_{i_{j_1}} = c_{i_{j_2}} = \dots = c \in T.$$

If $j < k$ and j belongs to the sequence $\{j_0, j_1, \dots\}$, then the edge from v_j to v_k is colored with the color c . It follows that the vertices

$$\{v_{j_0}, v_{j_1}, \dots\}$$

span a monochromatic subgraph of G . □

Ramsey's theorem has many generalizations. For example, we need not consider graphs.

Notation 3. Let X be a set. For each integer $n \geq 0$, we let $X^{(n)}$ denote the set of n -element subsets of X . Given a set T , we define a T -coloring of $X^{(n)}$ to be a function $f : X^{(n)} \rightarrow T$.

Example 4. When $n = 2$, a T -coloring of $X^{(n)}$ is just an edge coloring of the complete graph with vertex set X .

Theorem 1 admits the following generalization:

Theorem 5. *Let $n \geq 0$, let X be an infinite set, let T be a finite set, and let $f : X^{(n)} \rightarrow T$ be a T -coloring of $X^{(n)}$. Then there exists an infinite subset $Y \subseteq X$ which is "monochromatic": that is, for which the restriction of f to $Y^{(n)}$ is constant.*

Example 6. If $n = 0$, then the result is trivial: $X^{(n)}$ has only a single element, the map f carries it to a single color $c \in T$, and we can take Y to be any infinite subset of X .

If $n = 1$, then $X^{(1)} = X$. In this case, Theorem 5 asserts a version of the pigeonhole principle: any map f from an infinite set X to a finite set T must assume some value $c \in T$ infinitely often.

If $n = 2$, then Theorem 5 recovers the statement of Theorem 1.

Proof. We will proceed by induction on n (we have just seen that the theorem is true for $n \leq 2$; any of these cases will serve as a base case). Assume that $n > 0$. We now proceed as in the proof of Theorem 1. Choose an element $v_0 \in X$. We now define a map

$$g_0 : (X - \{v_0\})^{(n-1)} \rightarrow T$$

by the formula $g_0(S) = f(S \cup \{v_0\})$. Since $X - \{v_0\}$ is an infinite set, the inductive hypothesis implies that there exists an infinite subset $X_1 \subseteq X - \{v_0\}$ such that $g_0|_{X_1^{(n-1)}}$ takes some constant value $c_0 \in T$. Choose an element $v_1 \in X_1$, and define

$$g_1 : (X_1 - \{v_1\})^{(n-1)} \rightarrow T$$

by the formula $g_1(S) = f(S \cup \{v_1\})$. Since $X_1 - \{v_1\}$ is infinite, the inductive hypothesis implies that there is an infinite subset $X_2 \subseteq X_1 - \{v_1\}$ such that $g_1|_{X_2^{(n-1)}}$ is constant. Proceeding in this way, we obtain an infinite decreasing sequence of subsets

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots,$$

an infinite sequence of colors $c_0, c_1, \dots \in T$, and elements $v_i \in X_i$ satisfying the following condition: for every $(n - 1)$ -element subset $S \subseteq X_{i+1}$, we have

$$f(\{v_i\} \cup S) = c_i.$$

In particular, if $i_1 < i_2 < \dots < i_n$, we have

$$f(\{v_{i_1}, \dots, v_{i_n}\}) = c_{i_1}.$$

Since the set T is finite, some color $c \in T$ appears in the list

$$c_0, c_1, c_2, \dots$$

infinitely many times. We may therefore choose integers

$$i_0 < i_1 < i_2 < \dots$$

such that $c_{i_j} = c$ for all j . It follows that

$$\{v_{i_0}, v_{i_1}, \dots\}$$

is an infinite monochromatic subset of X . □

One might ask if some version of Theorem 5 also holds for finite sets. The answer is yes, and the proof given above can be adapted to the setting of finite sets. However, one can also deduce the finite version from the infinite version:

Theorem 7. *Let n and m be integers and let T be a finite set. Then there exists an integer C (depending on n , m , and T) with the following property: for every set X with at least C elements and every coloring $f : X^{(n)} \rightarrow T$, there exists a subset $Y \subseteq X$ of size m such that $f|_{Y^{(n)}}$ is constant.*

Proof. We now proceed by contradiction. Assume that no such integer C exists. For each k , let $\langle k \rangle = \{1, \dots, k\}$. Let us say that a coloring $f : \langle k \rangle^{(n)} \rightarrow T$ is *bad* if there does not exist a subset $Y \subseteq \langle k \rangle$ of size m such that f is constant on $Y^{(n)}$. We will denote the collection of all colorings of $\langle k \rangle^{(n)}$ by $Z(k)$, and the collection of all bad colorings by $Z_0(k) \subseteq Z(k)$. Note that we can regard $\langle k \rangle$ as a subset of $\langle k+1 \rangle$, so that every coloring of $\langle k+1 \rangle^{(n)}$ determines a coloring of $\langle k \rangle^{(n)}$. This gives us maps

$$\dots \rightarrow Z(2) \rightarrow Z(1) \rightarrow Z(0).$$

For each $k \geq 0$, we define a subset $Z_0(k) \subseteq Z(k)$ as follows: a coloring $f : \langle k \rangle^{(n)} \rightarrow T$ belongs to $Z_0(k)$ if and only if it extends to a bad coloring of $\langle l \rangle^{(n)}$, for each $l \geq k$. We claim that $Z_0(k)$ is nonempty. Assume otherwise. Then, for each element $f \in Z(k)$, there exists $l \geq k$ such that f does not extend to a bad coloring of $\langle l \rangle^{(n)}$. Since $Z(k)$ is finite, we can choose a single integer $l \geq k$ such that for every $f \in Z(k)$, f does not extend to a bad coloring of $\langle l \rangle^{(n)}$. This implies that $\langle l \rangle^{(n)}$ has no bad colorings, contradicting our assumption.

Let $f \in Z_0(k+1)$ and let g denote its image in $Z(k)$. We claim that $g \in Z_0(k)$: that is, g extends to a bad coloring of $\langle l \rangle^{(n)}$ for each $l \geq k$. This is clear, since f extends to a bad coloring of $\langle l \rangle^{(n)}$ for each $l > k$. But the converse is also true: if $g \in Z_0(k)$, then g is the image of a coloring $f \in Z_0(k+1)$. To prove this, let $\{f_1, \dots, f_a\}$ be the finite collection of all colorings of $\langle k+1 \rangle^{(n)}$ which extend g . If none of these belong to $Z_0(k+1)$, then we can choose an integer $l > k$ such that none of the colorings f_i extends to a bad coloring of $\langle l \rangle^{(n)}$. This means that g cannot be extended to a bad coloring of $\langle l \rangle^{(n)}$, contradicting our assumption that $g \in Z_0(k)$.

We therefore have a sequence of surjective maps

$$\dots \rightarrow Z_0(2) \rightarrow Z_0(1) \rightarrow Z_0(0).$$

Since $Z_0(0)$ is nonempty, we can choose a coloring $f_0 \in Z_0(0)$. Using the surjectivity, we can successively lift f_0 to elements $f_k \in Z_0(k)$. Taken together, the f_k determine a coloring

$$f_\infty : (\mathbf{Z}_{>0})^{(n)} \rightarrow T$$

of the infinite set $\mathbf{Z}_{>0}^{(n)}$. It follows from Theorem 5 that this coloring has an infinite monochromatic subset $Y \subseteq \mathbf{Z}_{>0}$. Write $Y = \{y_1 < y_2 < \dots\}$. Then $\{y_1, \dots, y_m\}$ is a monochromatic subset of $\langle y_m \rangle$ of size m , contradicting our assumption that f_{y_m} is bad. \square

The proof of Theorem 7 is a typical example of a *compactness* argument. It can be formulated naturally in the language of topology. Let Z denote the collection of all colorings $f_\infty : (\mathbf{Z}_{>0})^{(n)} \rightarrow T$. Since every such coloring is determined by its restriction to each of the finite subsets $\langle k \rangle^{(\infty)}$, we can identify Z with a subset of the product $\prod_{k \geq 0} Z(k)$. Each $Z(k)$ is a finite set, which we can endow with the discrete topology. We can regard $\prod_{k \geq 0} Z(k)$ as endowed with the product topology, and Z with the subspace topology. Using Tychanoff's theorem, we see that $\prod_{k \geq 0} Z(k)$ is compact. It is not hard to see that Z is a closed subset of $\prod_{k \geq 0} Z(k)$, and therefore also compact. For each $k \geq 0$, let C_k denote the subset of Z consisting of colorings whose restriction to $\langle k \rangle^{(n)}$ is bad. Then the C_k form a decreasing chain

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$$

of closed subsets of Z . If Theorem 7 is false, then there is a bad coloring of each $\langle k \rangle^{(n)}$. Extending this arbitrarily to a coloring of $\mathbf{Z}_{>0}^{(n)}$, we see that each C_i is nonempty. Using the compactness of Z , we deduce that $\bigcap_k C_k$ is nonempty. This gives a coloring of $\mathbf{Z}_{>0}^{(n)}$ whose restriction to each $\langle k \rangle^{(n)}$ is bad, which contradicts Theorem 5.