Math 155 (Lecture 25)

November 1, 2011

In Problem Set 7, you proved the following result:

Proposition 1. *Let A be a partially ordered set. If A has at least $mn + 1$ elements, then A has either a chain of length $m + 1$ or an antichain of length $n + 1$.*

In the situation of Proposition 1, we can associate to A a certain graph G. The vertices of G are the elements of A, and two vertices $x, y \in G$ are connected by an edge if and only if either $x < y$ or $y < x$. If $S \subseteq A$ is a set of vertices of G, then S is a chain (when regarded as a subset of A) if and only if it is a clique (when regarded as a subset of G): that is, if and only every pair of distinct elements of S are adjacent in G. A subset $S \subseteq A$ is an antichain (when regarded as a subset of A) if and only if it is an anticlique (when regarded as a subset of G): that is, if and only if no pairs of elements of S are adjacent in G. We can rephrase Proposition 1 as follows:

Proposition 2. *Let G be a graph which is constructed by the above procedure. If G has $mn + 1$ vertices, then G either contains a clique of size $m + 1$ or an anticlique of size $n + 1$.*

Of course, the graphs that arise from partially ordered sets are rather special. Nevertheless, there is a version of Proposition 2 which is true in general:

Theorem 3 (Ramsey's Theorem, First Version)**.** *Let m and n be integers. Then there exists an integer C with the following property: for every graph G with at least C vertices, G contains either a clique of size m or an anticlique of size n.*

Notation 4. Let m and n be integers. We let $R(m, n)$ denote the least integer C which satisfies the requirements of Theorem 3. The numbers $R(m, n)$ are called *Ramsey numbers.*

Remark 5. For any graph G, we can make a new graph G' with the same vertex set, where a pair of distinct edges $x, y \in G$ are adjacent in G if and only if they are *not* adjacent in G'. Note that a set of vertices forms a clique in G if and only if it forms an anticlique in G', and vice-versa. It follows that the considerations of Theorem 3 are symmetric in m and n. In particular, we have an equality of Ramsey numbers

$$R(m, n) = R(n, m).$$

Example 6. For every integer n, we have $R(0, n) = 0$ (every graph contains a clique of size zero).

Example 7. If $n > 0$, we have $R(1, n) = 1$ (since every nonempty graph contains a clique of size one).

Example 8. Let G be a graph. If G has any edges, then it contains a clique of size 2. Otherwise, G itself is an anticlique of size n, where n is the number of vertices of G. It follows that $R(2, n) = n$.

Proof of Theorem 3. We proceed by induction on m and n. If m or n is equal to zero, there is nothing to prove (we can take $C = 0$, by virtue of Example 6). Suppose therefore that $m, n > 0$. The inductive hypothesis implies that there exist integers C' and C'' with the following properties:

(a) If G is a graph with at least C' vertices, then G either contains a clique of size $m-1$ or an anticlique of size n.

(b) If G is a graph with at least C'' vertices, then G either contains a clique of size m or an anticlique of size n.

Now let G be any nonempty graph, and let $v \in G$ be a vertex. Let G_- be the subgraph of G spanned by those vertices which are adjacent to v, and G_+ the subgraph of G spanned by those vertices which are distinct from v and not adjacent to v.

For each graph H, let $|H|$ denote the number of vertices of H. If $|G_-| \geq C'$, then either G_- contains an anticlique of size n (in which case G does too), or G_- contains a clique of size $m-1$ (in which case G contains a clique of size m, obtained by adding the vertex v). Similarly, if $|G_+| \geq C''$, then G must contain either a clique of size m or an anticlique of size n. If neither of these conditions is satisfied, then we get

$$|G| = |G_-| + |G_+| + 1 \leq (C'-1) + (C''-1) + 1 = C' + C'' - 1 < C' + C''.$$

It follows that if $|G| \geq C' + C''$, then G contains either a clique of size m or an anticlique of size n. □

Remark 9. The proof of Theorem 3 gives the following inequality of Ramsey numbers: for $m, n \geq 2$ we have

$$R(m,n) \leq R(m-1,n) + R(m,n-1).$$

(If $m = n = 1$, this inequality fails, since $R(m-1,n) + R(m,n-1) = 0$, and in an empty graph we cannot choose a vertex v to start the proof of Theorem 3.

Example 10. Taking $m = n = 3$, we get an inequality

$$R(3,3) \leq R(2,3) + R(3,2) = 3 + 3 = 6.$$

That is, every graph of size 6 either contains a clique of size 3 or an anticlique of size 3. This is optimal: if G is the graph consisting of vertices and edges of a regular pentagon, then G does not contain a clique or anticlique of size 3.

We can use Remark 9 to get an upper bound for the Ramsey numbers $R(m,n)$:

Proposition 11. *Let $m, n \geq 1$ be positive integers. Then*

$$R(m,n) \leq \binom{m+n-2}{m-1} = \frac{(m+n-2)!}{(m-1)!(n-1)!}.$$

Proof. We proceed by induction on m and n. If m or n is equal to 1, then both sides are equal to 1 and there is nothing to prove. Let's therefore assume that $m, n \geq 2$. Combining Remark 9 with the inductive hypothesis, we get

$$R(m,n) \leq R(m-1,n) + R(m,n-1) \leq \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1}.$$

□

Remark 12. The inequality $R(m,n) \leq R(m-1,n) + R(m,n-1)$ is not sharp in general. For example, one can show that

$$R(3,4) = 9 < 10 = 4 + 6 = R(2,4) + R(3,3).$$

Remark 13. One can show that the Ramsey number $R(4,4)$ is equal to 18. The exact values of $R(n,n)$ are not known for $n \geq 5$. However, we do know that they grow very quickly with n: we will prove this in the next lecture.

Ramsey's theorem has many generalizations. Let us consider one.

Definition 14. Let G be a graph, and let $T = \{c_1, c_2, \ldots, c_t\}$ be a set of colors. An *edge coloring* of G is a function from the set of edges of G to the set T. Given an edge coloring of G, we will say that a set S of vertices of G is *monochromatic* if the edge coloring carries every edge joining two vertices of S to the same element $c \in T$. In this case, we say that S is *monochromatic with color c*.

Theorem 15 (Ramsey's Theorem, Several Color Version)**.** *Let $T = \{c_1, \ldots, c_t\}$ be a finite set, and suppose we are given a list of integers $n_1, n_2, \ldots, n_t$. Then there exists an integer C with the following property: for every edge coloring of the complete graph G with at least C vertices, there exists a set of vertices S of G which is monochromatic of some color c_i, and contains at least n_i elements.*

Remark 16. Theorem 3 is just the special case of Theorem 15 where $t = 2$. Note that to give a graph G with vertex set V is equivalent to giving an edge coloring of the complete graph with vertex set V, using the color set $T = \{ \text{in}, \text{out} \}$.

Notation 17. The least integer C satisfying the requirements of Theorem 15 is denoted by $R(n_1, n_2, \ldots, n_t)$.

Proof. As in the proof of Theorem 3, we proceed by induction on the integers $n_1, n_2, \ldots, n_t$. Let us assume that each n_i is ≥ 1 (otherwise, we can take $C = 0$).The inductive hypothesis implies that the Ramsey numbers

$$C_i = R(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_t)$$

are well-defined for $1 \leq i \leq t$.

Let G be a complete nonempty graph with at least

$$2 - t + \sum_{1 \leq i \leq t} C_i$$

vertices, with an edge coloring by the set T. Choose a vertex $v \in G$. For $1 \leq i \leq t$, let G_t be the graph spanned by those vertices w such that the edge joining v and w is colored with c_i. Then

$$1 + \sum_{1 \leq i \leq t} |G_i| = |G| \geq 2 - t + \sum_{1 \leq i \leq t} C_i,$$

so that

$$\sum_{1 \leq i \leq t} |G_i| > \sum_{1 \leq i \leq t} C_i - 1.$$

It follows that $|G_i| > C_i - 1$ for some i. Then either G_i contains a monochromatic subset of size n_j and color c_j for $j \neq i$ (in which case G does too), or G_i contains a monochromatic subset of size $n_i - 1$ and color c_i (in which case G contains a monochromatic subset of size n_i and color c_i, obtained by adding the vertex v). It follows that we can take $C = 2 - t + \sum_{1 \leq i \leq t} C_i$. $\square$

Remark 18. The proof of Theorem 15 gives the inequality

$$R(n_1, \ldots, n_t) \leq 2 - t + \sum_{1 \leq i \leq t} R(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_t)$$

provided that the sum on the right hand side is at least 1.