Math 155 (Lecture 22)

October 25, 2011

In this lecture, we will continue to study the Möbius function μ associated to a partially ordered set A. Our first goal is to understand how recover the Möbius function of a complicated partially ordered set in terms of the Möbiusfunction of simpler constituents.

Definition 1. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two partially ordered sets. Then the product $A \times B$ inherits a partial ordering, where we write

$$(a, b) \leq (a', b')$$

if and only if $a \leq_A a'$ and $b \leq_B b'$.

Proposition 2. *Let A and B be finite partially ordered sets, with Möbius functions μ_A and μ_B. Then the Möbius function of $A \times B$ is given by the formula*

$$\mu((a, b), (a', b')) = \mu_A(a, a')\mu_B(b, b').$$

Proof. As in the proof of Theorem **??**, it suffices to show that the matrix

$$[\mu_A(a, a')\mu_B(b, b')]_{(a,b),(a',b') \in A \times B}$$

is an inverse to the incidence matrix for the partially ordered set $A \times B$. Unwinding the definitions, we must show that for $a, a'' \in A$ and $b, b'' \in B$, the sum

$$\sum_{a' \geq_A a, b' \geq_B b} \mu_A(a', a'')\mu_B(b', b'')$$

is equal to one if $(a, b) = (a'', b'')$, and zero otherwise. By the distributive law, this sum is given by

$$(\sum_{a' \geq_A a} \mu_A(a', a''))(\sum_{b' \geq_B b} \mu_B(b', b'')).$$

Using the defining properties of μ_A and μ_B, we can write this as

$$(\begin{cases} 1 & \text{if } a = a'' \\ 0 & \text{otherwise.} \end{cases})(\begin{cases} 1 & \text{if } b = b'' \\ 0 & \text{otherwise.} \end{cases})$$

which is 1 if $(a, b) = (a'', b'')$, and 0 otherwise. □

Example 3. Let S be a finite set with n elements, and let $P(S)$ be the collection of all subsets of S, ordered by inclusion. Then $P(S)$ can be identified with the product

$$\prod_{s \in S} \{0 < 1\}$$

of n copies of the partially ordered set $A = \{0 < 1\}$. We have seen that the Möbius function μ_A of A is given by

$$\mu_A(i,j) = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i < j \\ 0 & \text{if } i > j. \end{cases}$$

Applying Proposition 2 repeatedly, we deduce that the Möbius function of $P(S)$ is given by

$$\mu_{P(S)}(I,J) = \begin{cases} (-1)^{|J-I|} & \text{if } I \subseteq J \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4. Let $(A, \leq)$ be a partially ordered set. We can define a new partial ordering $\leq'$ on A as follows:

$$(a \leq' b) \Leftrightarrow (b \leq a)$$

We will refer to the partial ordering $\leq'$ as the *opposite* of the original partial ordering $\leq$. We let A^{op} denote A, equipped with the opposite ordering.

Remark 5. If we understand the Möbius function μ_A of a partially ordered set A, it is easy to describe the Möbius function of its opposite A^{op}. Namely, we have

$$\mu_{A^{op}}(a,b) = \mu_A(b,a).$$

Now let $S = \{1, \ldots, n\}$ and let A be the *opposite* of the partially ordered set $P(S)$ (so that sets are ordered by reverse inclusion). Then the Möbius function of A is given by

$$\mu(I,J) = \begin{cases} (-1)^{|I-J|} & \text{if } I \supseteq J \\ \emptyset & \textit{otherwise.} \end{cases}$$

From this, we can recover the inclusion-exclusion principle. Let X be a finite set, equipped with subsets $X_1, X_2, \ldots, X_n \subseteq X$. For $J \subseteq \{1, \ldots, n\}$, we set

$$X_J = \bigcap_{i \in J} X_i \qquad X(J) = (\bigcap_{i \in J} X_i) \cap (\bigcap_{i \notin J} (X - X_i))$$

Then $|X_J| = \sum_{K \supseteq J} |X(J)|$. Applying Möbius inversion, we get

$$|X(J)| = \sum_{K \subseteq \{1,\ldots,n\}} \mu(K,J)|X_K| = \sum_{K \supseteq J} (-1)^{|K-J|}|X_K|.$$

In particular, if we take the set J to be empty, we recover the formula

$$|X - \bigcup_{1 \leq i \leq n} X_i| = \sum_{K \subseteq \{1,\ldots,n\}} (-1)^{|K|}|X_K|.$$

So far, we have discussed Möbiusinversion in the context of *finite* partially ordered sets. However, it is convenient to consider a mild generalization.

Definition 6. Let A be a partially ordered set. We will say that A is *locally finite* if, for every element $a \in A$, the set $A_{\leq a} = \{b \in A : b \leq a\}$ is finite.

Let A be a locally finite partially ordered set. For any pair of elements $a, b \in A$, there are only finitely many chains which start at a and end at b (because every such chain is contained in the finite partially ordered set $A_{\leq b}$). Consequently, we can still define the Möbiusfunction $\mu : A \times A \to \mathbf{Z}$ by the formula

$$\mu(a,b) = \sum_C (-1)^{l(C)},$$

where the sum is taken over all chains from a to b.

Proposition 7. *Let A be a locally finite partially ordered set, and let $f : A \to \mathbf{Z}$ be any function. Define a new function $g : A \to \mathbf{Z}$ by the formula*

$$g(b) = \sum_{a \leq b} f(a).$$

Then we can recover f by the formula

$$f(b) = \sum_{a} \mu(a, b) f(a).$$

(Note that this sum is well-defined, since $\mu(a, b) = 0$ unless a belongs to the finite set $A_{\leq b}$).

Proof. The general formula can be reduced to the case of finite partially ordered sets as follows. Fix an element $b \in A$, and let $\mu' : A_{\leq b} \times A_{\leq b} \to \mathbf{Z}$ be the restriction of μ. Then μ' agrees with the Möbius function for the finite partially ordered set $A_{\leq b}$ (note that if C is a chain from a to a' in $A_{\leq b}$, then C must be entirely contained in $A_{\leq b}$). Similarly let $f', g' : A_{\leq b} \to \mathbf{Z}$ be the restrictions of f and g. These functions satisfy

$$g'(c) = \sum_{a \leq c} f'(a),$$

so that

$$f'(c) = \sum_{a \in A_{\leq b}} \mu'(a, c) g'(a) = \sum_{a \leq c} \mu'(a, c) g'(a).$$

Taking $c = b$, we recover the formula

$$f(b) = \sum_{a \leq b} \mu(a, b) g(a).$$

□

Example 8. Let $\mathbf{Z}_{>0}$ be the set of positive integers, partially ordered by divisibility. Then $\mathbf{Z}_{>0}$ is locally finite (any divisor of n is $\leq n$, so every positive integer has only finitely many divisors). Let us compute the Möbiusfunction $\mu_{\mathbf{Z}_{>0}}$.

Fix an integer $n > 0$ with prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Let $X \subseteq \mathbf{Z}_{>0}$ be the set of divisors of n: namely, those integers of the form

$$p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}$$

where $f_i \leq e_i$ for $1 \leq i \leq k$. As a partially ordered set, X can be identified with the product

$$\prod_{1 \leq i \leq k} \{0, 1, \ldots, e_k\}.$$

Combining Proposition 2 with our understanding of the Möbius function of the factors, we see that the Möbius function μ_X of X is given by

$$\mu_X(\prod p_i^{f_i}, \prod p_i^{g_i}) = \prod_{1 \leq i \leq k} \begin{cases} 1 & \text{if } f_i = g_i \\ -1 & \text{if } f_i = g_i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we ahve

$$\mu_X(m, m') = \begin{cases} (-1)^j & \text{if } \frac{m'}{m} \text{ is a product of } j \text{ distinct primes.} \\ 0 & \text{otherwise.} \end{cases}$$

Since μ_X is just given by the restriction of $\mu_{\mathbf{Z}_{>0}}$ to X, we get

$$\mu_{\mathbf{Z}_{>0}}(m,n) = \begin{cases} (-1)^j & \text{if } \frac{n}{m} \text{ is a product of } j \text{ distinct primes.} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the integer $\mu_{\mathbf{Z}_{>0}}(m,n)$ depends only on the quotient $\frac{n}{m}$. It is therefore traditional to rewrite $\mu_{\mathbf{Z}_{>0}}$ as a function one variable. Let us say that an integer n is *square-free* if it is not divisible by the square of any prime. Define

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ is square-free} \\ 0 & \text{otherwise.} \end{cases}$$

We can then write

$$\mu_{\mathbf{Z}_{>0}}(m,n) = \begin{cases} \mu(\frac{n}{m}) & \text{if } m|n \\ 0 & \text{otherwise.} \end{cases}$$

The function $\mu : \mathbf{Z}_{>0} \to \mathbf{Z}$ constructed above often simply referred to as the *Möbiusfunction*. Applying Möbius inversion in this context gives the following:

Proposition 9. *Let $f : \mathbf{Z}_{>0} \to \mathbf{Z}$ be an arbitrary function, and define $g : \mathbf{Z}_{>0} \to \mathbf{Z}$ by the formula*

$$g(n) = \sum_{d|n} f(d).$$

Then we can recover f by the formula

$$f(n) = \sum_{d|n} g(d)\mu(\frac{n}{d}).$$

Example 10. Recall that Euler's ϕ-function $\phi : \mathbf{Z}_{>0} \to \mathbf{Z}$ assigns to each integer n the number of elements of the set $\{1, 2, \ldots, n\}$ which are relatively prime to n. Set $X = \{1, 2, \ldots, n\}$. For each $d|n$, let X_d denote the set of all elements $m \in X$ such that the greatest common divisor of m and n is d. The function $m \mapsto \frac{m}{d}$ induces a bijection from X_d to the subset of $\{1, 2, \ldots, \frac{n}{d}\}$ consisting of elements which are relatively prime to $\frac{n}{d}$. We therefore have $|X_d| = \phi(\frac{n}{d})$. Since X is given by the disjoint union of the X_d's, we obtain $n = \sum_{d|n} \phi(\frac{n}{d}) = \sum_{d|n} \phi(d)$. Applying Proposition 9, we get

$$\phi(n) = \sum_{d|n} d\mu(\frac{n}{d}),$$

which recovers the formula for ϕ that we deduced from the inclusion-exclusion principle.