Math 155 (Lecture 21)

October 25, 2011

Let A be a finite partially ordered set. The *incidence matrix* of A is the square matrix $I = [i_{a,b}]_{a,b \in A}$, where

$$i_{a,b} = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise.} \end{cases}$$

In the last lecture, we introduced the *Möbius function* of A. This is a function

$$\mu : A \times A \to \mathbf{Z}$$

with the following property: the matrix $[\mu(a,b)]_{a,b \in A}$ is an inverse of the incidence matrix I

Our first goal in this lecture is to give a more explicit description of the function μ. First, let us introduce a bit of notation.

Notation 1. Let A be a partially ordered set containing elements $a, b \in A$. We let $X_{a,b}$ denote the set of all chains $C \subseteq A$ containing a as a least element and b as a greatest element. In this case, we can write $C = \{a = x_0 < x_1 < \cdots < x_k = b\}$. We will refer to k as the *length* of C and denote it by $l(C)$, so that $l(C) = |C| - 1$.

Theorem 2. *Let A be a finite partially ordered set. The Möbius function $\mu : A \times A \to \mathbf{Z}$ is given by the formula*

$$\mu(a,b) = \sum_{C \in X_{a,b}} (-1)^{l(C)}.$$

Proof. Define $\lambda(a,b) = \sum_{C \in X_{a,b}} (-1)^{l(C)}$. To prove that $\lambda = \mu$, it will suffice to show that the matrix $M = [\lambda(a,b)]_{a,b \in A}$ is an inverse of the incidence matrix I. Since I is invertible, it will suffice to show that MI is the identity matrix. Unwinding the definitions, we must show that for $a, c \in A$, the sum

$$\sum_{b \in A} \lambda(b,c) \begin{cases} 1 & \text{if} a \leq b \\ 0 & \text{otherwise.} \end{cases}$$

is 1 if $a = c$ and zero otherwise. In other words, we wish to show

$$\sum_{b \geq a} \lambda(b,c) = \begin{cases} 1 & \text{if } a = c \\ 0 & \text{if } a \neq c. \end{cases}$$

Invoking the definition of $\lambda_{b,c}$, we can rewrite the right hand side as

$$\sum_{b \geq a} \sum_{C \in X_{b,c}} (-1)^{l(C)}.$$

This can be written as

$$\sum_{C \in Y_{a,c}} (-1)^{l(C)},$$

where $Y_{a,c}$ denotes the collection of all chains in A whose largest element is equal to c, and whose smallest element is $\geq a$.

If $a = c$, then $Y_{a,c}$ contains only a single chain $C = \{c\}$ of length 0, so this sum is equal to 1. Let us therefore assume that $a \neq c$, and prove that the sum is equal to zero. We divide the set $Y_{a,c}$ into two parts: let $Y_+ \subseteq Y_{a,c}$ be the collection of those chains which contain a, and let Y_- be the collection of those chains which do not. The construction $C \mapsto C \cup \{a\}$ determines a bijection from Y_- to Y_+ (the inverse bijection is given by $C \mapsto C - \{a\}$). We can therefore write

$$\sum_{C \in Y_{a,c}} (-1)^{l(C)} = \sum_{C \in Y_-} (-1)^{l(C)} + \sum_{C \in Y_-} (-1)^{l(C \cup \{a\})}.$$

On the right hand side, we can cancel the relevant terms pairwise to obtain 0. □

Corollary 3. *Let A be a finite partially ordered set and μ its Möbius function. Then μ has the following properties:*

(1) $\mu(a, a) = 1$ *for all* $a \in A$.

(2) *If* $a \not\leq b$, *then* $\mu(a, b) = 0$.

Proof. If $a = b$, then $X_{a,b}$ consists only of the chain $C = \{a\}$, so that $\sum_{C \in X_{a,b}} (-1)^{l(C)|} = 1$. This proves (1). To prove (2), note that if $a \not\leq b$ then $X_{a,b}$ is empty (there are no chains from a to b). □

Corollary 4. *Let A be a finite partially ordered set and μ its Möbius function. Then the definition of μ is local. That is, for each $a, b \in A$, the integer $\mu(a, b)$ depends only on the partially ordered set $\{c \in A : a \leq c \leq b\}$.*

Theorem 2 can be given a topological interpretation. To every partially ordered set A, one can associate a topological space $N(A)$, called the *nerve* of A. The space $N(A)$ is a simplicial complex, whose simplices are given by the chains of A. More precisely, we can construct $N(A)$ as follows:

- For each $a \in A$, add a vertex v_a.
- For each $a < b$ in A, add an edge $e_{a,b}$ from v_a to v_b.
- For each $a < b < c$, add a triangle with vertices v_a, v_b, and v_c, whose edges are given by $e_{a,b}$, $e_{b,c}$, and $e_{a,c}$.
- And so forth.

To be still more precise, if we choose an enumeration $A = \{a_1, \ldots, a_n\}$, then we can define $N(A)$ to be the subset of $\mathbb{R}^n$ consisting of those vectors $(t_1, t_2, \ldots, t_n)$ such that each $t_i \geq 0$, $\sum_{1 \leq i \leq n} t_i = 1$, and $\{a_i : t_i \neq 0\}$ is a chain of A.

To any finite simplicial complex Y, one can assign its *Euler characteristic* $\chi(Y)$. This is simply given by the alternating sum

$$\sum_{n \geq 0} (-1)^n s_n$$

where s_n denotes the number of n-simplices of Y. In particular, if A is a partially ordered set, we have

$$\chi(N(A)) = \sum_{\emptyset \neq C \subseteq A} (-1)^{l(C)},$$

where the sum is taken over all nonempty chains in A.

Now suppose we are given elements $a, b \in A$. Assume that $a < b$ (otherwise, the value of the Möbius function $\mu(a, b)$ is given by Corollary 3), and set $A_{>a}^{<b} = \{c \in A : a < c < b\}$. If C is a chain in A with greatest element b and least element a, then $C - \{a, b\}$ is a chain in $A^{<b}_{>a}$. Conversely, if $C \subseteq A^{<b}_{>a}$ is a chain, then $C \cup \{a, b\}$ is a chain in A with least element a and greatest element b. Using Theorem 2, we can write

$$\mu(a, b) = \sum_{C \in X_{a,b}} (-1)^{l(C)} = \sum_{D \subseteq A^{<b}_{>a}} (-1)^{l}(D \cup \{a, b\}) = (\sum_{\emptyset \neq D \subseteq A^{<b}_{>a}} (-1)^{l(D)}) - 1 = \chi(N(A^{<b}_{>a})) - 1.$$

Combining this with Corollary 3, we obtain the following:

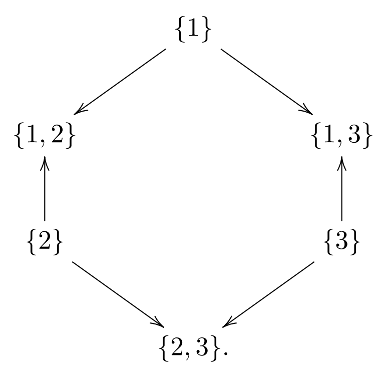
Proposition 5. *Let A be a finite partially ordered set. Then the Möbius function $\mu : A \times A \to \mathbf{Z}$ is given by*

$$\mu(a, b) = \begin{cases} 1 & \text{if } a = b \\ \chi(N(A^{<b}_{>a})) - 1 & \text{if } a < b \\ 0 & \text{otherwise.} \end{cases}$$

Example 6. Suppose we elements $a < b$ of A which are *adjacent*, in the sense that there do not exist any elements c with $a < c < b$. Then $A^{<b}_{>a}$ is empty, so $\chi(N(A^{<b}_{>a})) = 0$, and $\mu(a, b) = -1$.

Example 7. Let A be the collection of all subsets of the set $\{1, 2\}$, ordered by inclusion. Let $a = \emptyset$ and $b = \{1, 2\}$ be the least and greatest elements of A, respectively. Then $A^{<b}_{>a}$ is the collection of one-element subsets of $\{1, 2\}$, which is a two-element antichain. Then the nerve $N(A^{<b}_{>a})$ consists of two points (with the discrete topology). We get $\chi(N(A^{<b}_{>a})) = 2$, so that $\mu(a, b) = 1$.

Example 8. Let A be the collection of all subsets of the set $\{1, 2, 3\}$, ordered by inclusion. Set $a = \emptyset$ and $b = \{1, 2, 3\}$. The nerve of $A^{<b}_{>a}$ is a one-dimensional simplicial complex depicted in the diagram



Topologically, this is a circle. It has Euler characteristic 0 (since it has 6 vertices and 6 edges), so we get $\mu(a, b) = -1$.

Examples 7 and 8 can be generalized. Let A be the collection of subsets of the set $\{1, \ldots, n\}$. If we remove the least and greatest elements $a, b \in A$, we obtain a new partially ordered set A_0. One can show that $N(A_0)$ is a sphere of dimension $n - 2$, and therefore has Euler characteristic $1 + (-1)^n$. It follows that $\mu(a, b) = (-1)^n$. However, we would like to prove this more directly, without making a digression into topology.