Math 155 (Lecture 20)

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Let X be a partially ordered set. Let $C \subseteq X$ be a chain, and $A \subseteq X$ an antichain. Then $C \cap A$ is both a chain and an antichain: it follows that $|C \cap A| \leq 1$.

Suppose that X is partitioned into antichains $A_1, \ldots, A_m$. Then every chain $C \subseteq X$ can contain at most one element of each A_i. It follows that $|C| \leq m$. Similarly, if we partition X into chains $C_1, \ldots, C_n$, then every antichain $A \subseteq X$ must satisfy $|A| \leq n$. Our first goal in this lecture is to show that both of these bounds are sharp.

Theorem 1. *Let X be a finite partially ordered set and let $m \geq 0$. The following conditions are equivalent:*

(1) *Every chain in X has size $\leq m$.*

(2) *The set X can be written as a union of m antichains.*

Theorem 2 (Dilworth)**.** *Let X be a finite partially ordered set and let $n \geq 0$. The following conditions are equivalent:*

(1) *Every antichain in X has length $\leq n$.*

(2) *The set X can be written as a union of n chains.*

Though the statements of Theorems 1 and 2 are analogous, Theorem 1 is considerably easier to prove. Let us begin with its proof.

Definition 3. Let X be a finite partially ordered set. The *height* of an element $x \in X$ is the length of the largest chain

$$x_0 < x_1 < \cdots < x_h = x$$

ending in x.

Example 4. An element $x \in X$ is minimal if and only if it has height 0.

We have already established the implication (2) $\Rightarrow$ (1) of Theorem 1. To prove the converse, suppose that every chain in X has length $\leq m$. We wish to show that X can be partitioned into m antichains. Note that every element of X has height $< m$. We may therefore write $X = X_0 \cup X_1 \cup \cdots \cup X_{m-1}$, where X_i denotes the subset of X consisting of elements of height i. To complete the proof, it will suffice to show that each X_i is an antichain. Suppose that $x, y \in X_i$. Then x has height i, so there exists a chain

$$x_0 < x_1 < \ldots < x_h = x.$$

If $x < y$, then we have a chain $x_0 < x_1 < \ldots < x_h < y$, contradicting our assumption that y has height i. It follows that X_i is an antichain, and Theorem 1 is proved.

The proof of Theorem 2 is a bit trickier. Once again, we have already proven that (2) $\Rightarrow$ (1). We wish to prove the converse: in other words, we wish to show that X can be partitioned into n chains, where n is the width of X (that is, the largest size of an antichain in X).

We will proceed by induction on the cardinality of X. If X is empty, then the theorem is trivial. Let us therefore assume that X is nonempty. Let n be the width of X and choose a chain $C \subseteq X$ whose size is as large as possible. Let c_- denote the smallest element of C and c_+ the largest element of C (these elements might be the same). There are two cases to consider:

(a) Suppose that $X - C$ has width $< n$. It follows from the inductive hypothesis that we can partition $X - C$ into chains $C_1, \ldots, C_m$ for $m < n$. Then $X = C_1 \cup \cdots \cup C_m \cup C$ can be partitioned into $m+1 \leq n$ chains.

(b) Suppose that $X - C$ has width n. Choose an antichain $\{a_1, \ldots, a_n\} \subseteq X - C$. Set

$$X_+ = \{x \in X : [(\exists 1 \leq i \leq n)[x \geq a_i]\}$$

$$X_- = \{x \in X : [(\exists 1 \leq i \leq n)[x \leq a_j]\}.$$

Note that $X = X_- \cup X_+$: if there were to exist an element $x \in X$ belonging to neither X_- or X_+, then $\{a_1, \ldots, a_n, x\}$ would be an antichain of size $n+1$ in X.

Moreover, we have $X_- \cap X_+ = \{a_1, \ldots, a_n\}$: if $x \in X_+ \cap X_-$, then we can write

$$a_i \leq x \leq a_j$$

for some $1 \leq i \leq j \leq n$. Since $\{a_1, \ldots, a_n\}$ is an antichain, we conclude that $i = j$, so that $x = a_i \in \{a_1, \ldots, a_n\}$.

Finally, we claim that X_- and X_+ are strictly smaller than X. In fact, we claim that X_+ does not contain the element $c_- \in C$. Otherwise, there exists $1 \leq i \leq n$ such that $a_i \leq c_-$. Since $a_i \in X - C$, we have $a_i \neq c_-$, so that $C \cup \{a_i\}$ is a chain strictly larger than C. This contradiction shows that $c_- \notin X_+$, and similarly $c_+ \notin X_-$.

Since $|X_+|, |X_-| < |X|$, we can apply the inductive hypothesis to each. Each has width n (since X_- and X_+ are contained in X, they can have width at most n; since each contains the antichain $\{a_1, \ldots, a_n\}$, it has size exactly n). We therefore deduce that X_- can be partitioned into chains $C_{-,1}, C_{-,2}, \ldots, C_{-,n}$. Since each of these chains can contain at most one of the elements a_j, we conclude that each $C_{-,i}$ contains exactly one of the elements a_j. Reindexing, we may assume that $a_i \in C_{-,i}$ for each i. Note that a_i must be a greatest element of $C_{-,i}$. Otherwise we have $a_i \leq x$ for $x \in C_{-,i}$, in which case (since $x \in X_-$) we get $a_i \leq x \leq a_j$ for some j, which forces $i = j$ and therefore $x = a_i$.

The same argument gives us a partition of X_+ into chains $C_{+,1}, \ldots, C_{+,n}$ such that each $C_{+,i}$ contains a_i as a least element. Then each of the sets $C_i = C_{-,i} \cup C_{+,i}$ is a chain, and we have $X = C_1 \cup \cdots \cup C_n$. This completes the proof of Theorem 2

We now turn to different topic: an idea called *Möbius inversion*. Let's begin with a brief review of the inclusion-exclusion principle. Let X be a set equipped with subsets $X_1, \ldots, X_n \subseteq X$. For each $J \subseteq \{1, \ldots, n\}$, set

$$X_J = \bigcap_{i \in J} X_i \qquad X(J) = (\bigcap_{i \in J} X_i) \cap (\bigcap_{i \notin J} (X - X_i))$$

Then X_J is given by the disjoint union of the sets $X(K)$, where K ranges over subsets of $\{1, \ldots, n\}$ containing J. We therefore have

$$|X_J| = \sum_{K \supseteq J} |X(K)|. \tag{1}$$

The inclusion-exclusion principle gives a formula for $X(\emptyset) = X - \bigcup_i X_i$ in terms of the sizes $|X_J|$; namely the formula

$$|X(\emptyset)| = \sum_K (-1)^{|K|} |X_K|.$$

This can be regarded as a special case of the following more general situation:

Question 5. Let A be a finite partially ordered set, and suppose we are given integers $\{m_a\}_{a \in A}$. For $a \in A$, set

$$n_a = \sum_{b \leq a} m_a.$$

How can we recover the integers m_a from the integers n_a?

Remark 6. To see the inclusion-exclusion principle is a special case of Question 5, we take A to be the collection of all subsets of $\{1, \ldots, n\}$, partially ordered by *reverse* inclusion. For $J \in A$, set $m_J = |X(J)|$, so that equation 1 gives

$$n_J = \sum_{K \supseteq J} m_J = |X_J|.$$

Proposition 7. *Let A be a finite partially ordered set. Then there exists a function $\mu : A \times A \to \mathbf{Z}$ with the following property: given any collection of integers $\{m_a\}_{a \in A}$, if we set*

$$n_a = \sum_{b \leq a} m_a,$$

then $m_a = \sum_{b \in A} \mu(b, a) n_a$.

The function $\mu : A \times A \to \mathbf{Z}$ is called the *Möbius function* of the partially ordered set A.

Proof. In the last lecture, we proved that the partial ordering on A can be refined to a linear ordering. That is, we can write $A = \{a_1, a_2, \ldots, a_k\}$, where $a_i \leq a_j$ only when $i = j$. Let M be the k-by-k matrix given by

$$M_{i,j} = \begin{cases} 1 & \text{if } a_i \leq a_j \\ 0 & \text{otherwise} \end{cases}$$

We can then regard a collection of integers $\{m_a\}_{a \in A}$ as a column vector

$$(m_{a_1}, m_{a_2}, \ldots, m_{a_k}).$$

Then

$$n_{a_j} = \sum_{a_i \leq a_j} m_{a_i} = \sum_{1 \leq i \leq k} M_{i,j} m_{a_i},$$

so that $\{n_a\}_{a \in A}$ is the column vector obtained by acting by the matrix $M_{i,j}$. To have

$$m_a = \sum_{b \in A} \mu(b, a) n_a,$$

we want the matrix $\{\mu(a_i, a_j)\}$ to be an inverse of the matrix $M_{i,j}$. To guarantee that this matrix exists, it suffices to know that the determinant of the matrix $M_{i,j}$ is equal to 1. This follows from the fact that $M_{i,j}$ is a lower triangular matrix with 1's along the diagonal. $\square$

Of course, Proposition 7 by itself does not recover the inclusion-exclusion principle. It tells us only that there exists a formula of the form

$$|X - \bigcup_i X_i| = \sum_J c_J |X_J|,$$

for some integers c_J (determined by the Möbius function of the partially ordered set $P(\{1, \ldots, n\})$). To see that $c_J = (-1)^{|J|}$, we need to know something about the function μ appearing in Proposition 7. We will take this up in the next lecture.