Math 155 (Lecture 2)

September 1, 2011

The goal of this lecture is to introduce a basic technique in combinatorics: the method of generating functions. Let begin with a simple counting problem.

Question 1. How many ways are there to tile a 2-by-n board with dominoes?

The answer, of course, is an integer which depends on n. Let's denote that integer by T_n.

Example 2. We have $T_1 = 1$; the unique tiling may be depicted as

$$\sqcap$$

$$\sqcup$$

Example 3. We have $T_2 = 2$. The two tilings may be depicted as

$$\sqcap \quad \sqcap \qquad\qquad \sqsubset \quad \sqsupset$$

$$\sqcup \quad \sqcup \qquad\qquad \sqsubset \quad \sqsupset$$

Example 4. We have $T_3 = 3$. The three tillings are given by

$$\sqcap \quad \sqcap \quad \sqcap \qquad\qquad \sqcap \quad \sqsubset \quad \sqsupset \qquad\qquad \sqsubset \quad \sqsupset \quad \sqcap$$

$$\sqcup \quad \sqcup \quad \sqcup \qquad\qquad \sqcup \quad \sqsubset \quad \sqsupset \qquad\qquad \sqsubset \quad \sqsupset \quad \sqcup$$

Example 5. We have $T_4 = 5$. The five tilings may be depicted

$$\sqcap \quad \sqcap \quad \sqcap \quad \sqcap \qquad\qquad \sqsubset \quad \sqsupset \quad \sqsubset \quad \sqsupset$$

$$\sqcup \quad \sqcup \quad \sqcup \quad \sqcup \qquad\qquad \sqsubset \quad \sqsupset \quad \sqsubset \quad \sqsupset$$

$$\sqcap \quad \sqcap \quad \sqsubset \quad \sqsupset \qquad\qquad \sqsubset \quad \sqsupset \quad \sqcap \quad \sqcap$$

$$\sqcup \quad \sqcup \quad \sqsubset \quad \sqsupset \qquad\qquad \sqsubset \quad \sqsupset \quad \sqcup \quad \sqcup$$

and

$$\sqcap \quad \sqsubset \quad \sqsupset \quad \sqcap$$

$$\sqcup \quad \sqsubset \quad \sqsupset \quad \sqcup$$

To say something about the integers T_n in general, we observe that they satisfy a *recurrence relation*. We can derive this relation as follows. Let X_n be the *set* of all domino tilings of a 2-by-n board, so that $T_n = |X_n|$. We can partition X_n into a pair of subsets X_n^+ and X_n^- as follows:

- Let X_n^+ be the set of all tilings where the leftmost column is filled with a vertically placed domino. To describe an element of X_n^+, we just need to give a domino tiling of the remaining 2-by-$(n-1)$ board, so that $|X_n^+| = T_{n-1}$.
- Let X_n^- be the complement of X_n^+. Then X_n^- consists of those domino tilings of a 2-by-n board where the far left is tiled by a pair of horizontally placed dominoes. To specify an element of X_n^-, we just need to describe the tiling of the remaining 2-by-$(n-2)$ board, so that $|X_n^-| = T_{n-2}$.

From the analysis, we deduce the following recurrence relation:

$$T_n = |X_n| = |X_n^+| + |X_n^-| = T_{n-1} + T_{n-2}.$$

From this we can quickly compute other values of T:

$$T_5 = T_4 + T_3 = 5 + 3 = 8$$

$$T_6 = T_5 + T_4 = 8 + 5 = 13$$

$$\dots$$

Remark 6. It is convenient to adopt the convention that $T_0 = 1$: that is, there is a unique tiling of the empty board. This is consistent with our recurrence relation, which dictates $T_0 = T_2 - T_1 = 2 - 1$.

In other words, $\{T_n\}_{n\geq 0}$ is the famous *Fibonacci sequence*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$$

Let us now go further and find a *closed-form* expression for the sequence $\{T_n\}_{n\geq 0}$. For this, we introduce the following formal expression

$$F(x) = \sum_{n\geq 0} T_n x^n.$$

Here $F(x)$ is a power series in a variable x. This power series actually converges for x sufficiently small, but we do not really need to know this: we will treat $F(x)$ as a formal algebraic expression.

Let us now see what our recurrence relation says about $F(x)$. We have

$$\begin{aligned}
F(x) &= \sum_{n\geq 0} T_n x^n \\
&= 1 + x + \sum_{n\geq 2} T_n x^n \\
&= 1 + x + \sum_{n\geq 2} (T_{n-1} + T_{n-2}) x^n \\
&= 1 + x + x(\sum_{n\geq 2} T_{n-1} x^{n-1}) + x^2(\sum_{n\geq 2} T_{n-2} x^{n-2}) \\
&= 1 + x + x(\sum_{n\geq 1} T_n x^n) + x^2(\sum_{n\geq 2} T_n x^n) \\
&= 1 + x + x(F(x) - 1) + x^2 F(x) \\
&= 1 + (x + x^2) F(x).
\end{aligned}$$

Solving this equation, we get

$$F(x) = \frac{1}{1 - x - x^2}.$$

Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio, so that the roots of the quadratic $1 - x - x^2$ are given by $-\phi$ and $\phi - 1 = \frac{1}{\phi}$. We therefore have $1 - x - x^2 = (1 + \frac{x}{\phi})(1 - \phi x)$. We therefore have a partial fraction decomposition of $F(x)$, given by

$$F(x) = \frac{\lambda}{1 + \frac{x}{\phi}} + \frac{\mu}{1 - \phi x},$$

where the scalars λ and μ satisfy

$$\lambda(1 - \phi x) + \mu(1 + \frac{x}{\phi}) = 1.$$

Extracting coefficients of x, we deduce

$$-\lambda\phi + \frac{\mu}{\phi} = 0,$$

so that $\mu = \phi^2\lambda$. Taking constant terms, we get $\lambda + \mu = \lambda(1 + \phi^2) = 1$, so that $\lambda = \frac{1}{1+\phi^2}$. Thus

$$\begin{aligned} F(x) &= \frac{1}{1+\phi^2}\frac{1}{1+\frac{x}{\phi}} + \frac{\phi^2}{1+\phi^2}\frac{1}{1-\phi x} \\ &= \frac{1}{1+\phi^2}\sum_{n \geq 0}(\frac{-1}{\phi})^n x^n + \frac{\phi^2}{1+\phi^2}\sum_{n\geq 0}\phi^n x^n \end{aligned}$$

Extracting the coefficient of x^n, we obtain the closed form expression

$$T_n = \frac{1}{1+\phi^2}(\frac{-1}{\phi})^n + \frac{\phi^2}{1+\phi^2}\phi^n$$

Remark 7. The expression $\frac{-1}{\phi}$ has absolute value less than one. For n large, the first term in our expression for T_n becomes very small, so that

$$T_n \approx \frac{\phi^{n+2}}{1+\phi^2}.$$

Let us now describe consider another counting problem which we can approach using generating functions. Let n and k be nonnegative integers. We let $\left\{ {n \atop k} \right\}$ denote the number of ways to partition the set $\{1, \ldots, n\}$ into k (unlabelled) nonempty subsets. The numbers $\left\{ {n \atop k} \right\}$ are called *Stirling numbers of the second kind*. For example, we have $\left\{ {4 \atop 2} \right\} = 7$, because we can decompose $\{1, 2, 3, 4\}$ into two subsets in precisely seven ways:

$$\{1,2\} \cup \{3,4\} \quad \{1,3\} \cup \{2,4\} \quad \{1,4\} \cup \{2,3\}$$

$$\{1\} \cup \{2,3,4\} \quad \{2\} \cup \{1,3,4\} \quad \{3\} \cup \{1,2,4\} \quad \{4\} \cup \{1,2,3\}.$$

The following is a prototypical problem in enumerative combinatorics:

- Give a closed-form expression for the Stirling numbers $\left\{ {n \atop k} \right\}$.

Let us illustrate an algebraic approach to this problem, at least for small values of k. We begin by showing that the Stirling numbers $\left\{ {n \atop k} \right\}$ satisfy a *recurrence relation*: for $n > 0$, we have

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}.$$

To see this, let S be the collection of all ways to partition the set $\{1, \ldots, n\}$ into k nonempty subsets. We divide S into two subsets:

- Let S_0 be the collection of all partitions of $\{1,\ldots,n\}$ into a collection of subsets which includes $\{n\}$. To give an element of S_0, we need to specify the *other* $k-1$ subsets, which partition the set $\{1,\ldots,n-1\}$. It follows that $|S_0| = \left\{ {n-1 \atop k-1} \right\}$.
- Let S_1 be the collection of all partitions of $\{1,\ldots,n\}$ into a collection of subsets which does not include $\{n\}$. Every such partition determines a partition of $\{1,\ldots,n-1\}$ into k subsets. We can then recover our original partition by adding the element n to any of these k subsets. It follows that

$$|S_1| = k\left\{ {n-1 \atop k} \right\}.$$

Putting these observations together, we obtain the formula

$$\left\{ {n \atop k} \right\} = |S| = |S_0| + |S_1| = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}.$$

To apply this recurrence relation, let us again form a generating function. For every integer k, we define a function $F_k(x)$ by the formula

$$F_k(x) = \left\{ {0 \atop k} \right\} + \left\{ {1 \atop k} \right\}x + \left\{ {2 \atop k} \right\}x^2 + \left\{ {3 \atop k} \right\}x^3 + \cdots$$

Applying our recurrence relation, we obtain

$$\begin{aligned} F_k(x) &= \sum \left\{ {n \atop k} \right\} x^n & (1) \\ &= \sum \left\{ {n-1 \atop k-1} \right\} x^n + k \sum \left\{ {n-1 \atop k} \right\} x^n & (2) \\ &= xF_{k-1}(x) + kxF_k(x). & (3) \end{aligned}$$

Manipulating this equation, we obtain

$$(1-kx)F_k(x) = xF_{k-1}(x)$$

$$F_k(x) = \frac{x}{1-kx}F_{k-1}(x).$$

Note that $F_0(x) = 1$ (there is no way to partition an empty set into a nonempty number of subsets). We therefore obtain

$$F_1(x) = \frac{x}{1-x} = x + x^2 + x^3 + \cdots$$

(This is rather obvious: for every $n \geq 1$, there is precisely one way to partition the set $\{1,\ldots,n\}$ into one nonempty subset.)

$$F_2(x) = \frac{x}{1-x}\frac{x}{1-2x} = \frac{x^2}{(1-x)(1-2x)}.$$

$$F_3(x) = \frac{x}{1-x}\frac{x}{1-2x}\frac{x}{1-3x} = \frac{x^3}{(1-x)(1-2x)(1-3x)}$$

and so forth. Let us use this to find a closed form expression for the Stirling numbers $\left\{ {n \atop 2} \right\}$. We have a partial fraction decomposition

$$\begin{aligned} F_2(x) &= x^2\left(\frac{1}{(1-x)(1-2x)}\right) \\ &= x^2\left(\frac{-1}{1-x} + \frac{2}{1-2x}\right) \\ &= \sum_{m \geq 0} (-1 + 2^{m+1})x^{m+2}. \end{aligned}$$

For $n \geq 2$, we can extract the coefficient of x^n to obtain the formula

$$\left\{ {n \atop 2} \right\} = 2^{n-1} - 1.$$

Exercise 8. Check this directly from the definition.